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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, découvrira plutôt les véritables loix de la Nature dans tout leur éclat, et on y trouvera des raisons encore plus fortes, d'en admirer la beauté et la simplicité. EULER

Ceux qui aiment l'Analyse verront avec plaisir la Mechanique en devenir une nouvelle branche ... LAGRANGE

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Exact Theory of Stress and Strain in Rods and Shells

J. L. ERICKSEN & C. TRUESDELL

Dedicated to the memory of E. & F. COSSERAT

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*1. Historical introduction**. The theory of *bending of elastic rods* was initiated by JAMES BERNOULLI in 1691. In his work, as well as in a paper of VARIGNON from 1702, the basic principles of statics as applicable to a rod are distinguished from the particular hypothesis of elasticity, but this separation has often been neglected by later writers. An exception is furnished by a remarkable memoir of EULER [1771], where the fully general *statical equations for a rod* bent in its own plane are derived. While EULER developed also a theory of bending of skew rods, all the researches in the seventeenth and eighteenth centuries remained primitive from lack of an adequate description of *strain* in a rod.

Such a description was initiated by ST. VENANT [1843] [1845], who introduced the concept of *twist* and the principal torsion-flexure axes (*cf.* also BINET [1844]). Other attempts to formulate a theory of strain in a rod were made by KIRCHHOFF [1859, § 2] [1876, Vorl. 28, § 2] and CLEBSCH [1862, §§ 48–49, 55]. All this early work employs more or less hidden approximations and is difficult to follow with confidence. The first straightforward analysis is that of LOVE [1893, 2, § 233]. While a great advance beyond its predecessors, his work is closely bound to the concepts of small deformation and linear elasticity.

That six equations are needed to express equilibrium of a *bent and twisted rod* was first remarked by ST. VENANT [1843], but he did not succeed in obtaining them without simplifying hypotheses. The exact general equations were given in principle, but obscurely, by KIRCHHOFF [1859, § 3], explicitly by CLEBSCH [1862, § 50]. These and other early treatments are difficult to follow, sometimes imparting the impression that some approximation is made. In fact, as was noted by BASSET [1895, § 2], the statical equations referred to the actual position of the rod are exact; no question of approximation arises unless we attempt to refer the equations to a configuration assumed by the rod prior to its being subjected to the load under which it is in equilibrium.

The theory of *curved elastic shells* is of more recent date. In the pioneer paper of LOVE [1888], neither the statical equations nor the description of strain is disentangled from the special elastic hypotheses and simplifying approximations. The exact statical equations were obtained by LAMB [1890]. The descriptions of strain used by classical writers are rather intuitive than proven and egress little, if at all, from the domain of infinitesimals.

The flood of papers on rods and shells brought on by the twentieth century has added little to principle. When not erroneous, in respect to the foundations they are for the most part repetitions or amplifications, sometimes more concisely expressed, of older ideas. Even works devoted expressly to finite deflections turn upon retaining additional terms in series expansions, *etc.*

Exceptions to the foregoing summary are the paper of HAY [1942] on rods and that of SYNGE & CHIEN [1941] (*cf.* also CHIEN [1944]) on shells. These authors obtain correct and general descriptions of *finite strain*, independent of the elastic hypotheses adopted in the later parts of their papers. While they employ tensorial methods to describe the strain, they follow the classical custom of selecting special co-ordinates. Not only are these co-cordinates not necessarily the best

* A full history of the early theories of rods will be included in *The rational mechanics of flexible or elastic bodies*, 1638–1788, Editor's introduction to L. Euleri Opera omnia II **11**, forthcoming.

suited to the statement of each special problem, but also in the theory of finite deformation they give rise to an unsymmetrical if not formidably elaborate set of equations. In their treatment of *stress*, however, SYNGE & CHIEN by using *fully general co-ordinates* obtain a most elegant and compact formal statement.

A new idea, supple in application to a variety of mechanical theories and formalisms, was proposed by DUHEM [1893, 1, Ch. II]: A body is to be regarded as a collection not only of points but also of *directions associated with the points*. These vectors, which we shall call the *directors* of the body, are susceptible of rotations and stretches *independent* of the deformation of material elements. Such a model of an *oriented body* should include, among other possibilities, representation of a molecular concept in which the molecules have internal structure. But also, as E. & F. COSSERAT [1907] [1908] [1909] remarked, in one and two dimensions it serves admirably to represent the twisting of rods and shells in addition to their bending. The COSSERATS constructed descriptions of this kind. These descriptions are set in a single rectangular Cartesian co-ordinate system and are formally elaborate from lack of any direct notation; moreover, they are interjected in a long plea for a kind of generalized elastic strain energy which the COSSERATS called "Euclidean action". Except for an exposition by SUDRIA [1935]*, this profound work of the COSSERATS has attracted no attention. Their considerations, while seeming to us incomplete as well as cumbrous, suggest the general theory of strain we now formulate.

2. The nature and plan of this work. For the notable successes of the three-dimensional theory of finite elastic deformation, as well as more general theories of recent date, an exact description of stress and strain was a necessary preliminary. Similarly precise and general theories of finite deformation of shells and rods might have been expected to follow upon these now well understood and developed branches of mechanics. That they have not, reflects, in our opinion, the lack of *a precise and general description of stress and strain in rods and shells.* Such a description, divorced from any constitutive assumption intended to describe the elastic or plastic response of the material, we here construct.

First we give some mathematical preliminaries and a sketch of those properties of an oriented body which are independent of the number of dimensions. Thereafter follow the theories of strain of rods and shells:

1. Strain of position
   - α. Intrinsic theory
   - β. Imbedding theory
2. Strain of orientation.

The intrinsic theory, based on the first fundamental form, coincides with the classical theory of strain; the imbedding theory rests on the curvatures and is known from differential geometry. A brief restatement of these results is included for completeness. The bulk of the paper concerns the strain of orientation. Here again the analysis is divided into two parts: 1°, differential description of the

* In his § 9 SUDRIA notes an error in the COSSERATS' argument and gives a different proof of invariance. We do not investigate the details, since our treatment is simpler and more general from the start. We note that in his Ch. II SUDRIA analyses time rates, a topic not included in the present work.

undeformed oriented body, and 2°, differential description of the deformation. Proofs of invariance and completeness are given. Despite our acknowledged debt to the Cosserats, we consider this analysis new in the main; in particular, its approach differs fundamentally from those of Hay and of Synge & Chien, whose results we pause to include as special cases for specially selected co-ordinates.

As is well known, it is far easier to obtain an exact description of stress than of finite strain, and even for rods and shells the only improvements to be expected here are in method. For rods we give essentially Heun's [1913, § 19] compaction of the Cosserats' argument. For shells we obtain the general equations of Synge & Chien by a briefer method made possible by use of the concept of double tensor field and total covariant derivative, explained in § 3 and used in our theory of the strain of shells. While we regard our material on stress as essentially classical, we include it not only for completeness but also for method. As far as stress is concerned, our line of argument, essentially that used by Heun for rods, is uniformly valid for a surface of p dimensions imbedded in a Euclidean space of $p+q$ dimensions.

The reader is assumed familiar with the classical notions of stress and strain in three dimensions as explained, *e.g.*, by Truesdell [1952], but no prior familiarity with theories of rods and shells is needed.

Part I. Preliminaries

3. Some mathematical tools. We consider two spaces, one with real co-ordinates X^K, $K=1, 2, \ldots, N$, or simply $\boldsymbol{X}$, and co-ordinate transformations $\boldsymbol{X}'=\boldsymbol{X}'(\boldsymbol{X})$, and the second with real co-ordinates x^k, $k=1, 2, \ldots, n$, or $\boldsymbol{x}$, and transformations $\boldsymbol{x}'=\boldsymbol{x}'(\boldsymbol{x})$. A set of functions $T^{K\ldots Pk\ldots p}_{Q\ldots Rq\ldots r}$ of the pair of points $\boldsymbol{X}$ and $\boldsymbol{x}$ is a *double tensor* field if it transforms according to the rule

$$\begin{aligned} T'^{K\ldots Pk\ldots p}_{Q\ldots Rq\ldots r} = T^{S\ldots Us\ldots u}_{V\ldots Wv\ldots w}\frac{\partial X'^K}{\partial X^S}\cdots\frac{\partial X'^P}{\partial X^U}\,\frac{\partial X^V}{\partial X'^Q}\cdots\frac{\partial X^W}{\partial X'^R}\times \\ \times\frac{\partial x'^k}{\partial x^s}\cdots\frac{\partial x'^p}{\partial x^u}\,\frac{\partial x^v}{\partial x'^q}\cdots\frac{\partial x^w}{\partial x'^r}. \end{aligned} \tag{3.1}$$

A familiar example of a double tensor field is the vector connecting two points in a Euclidean space: If p^k and P^K are the position vectors of the two points in the co-ordinates selected at $\boldsymbol{x}$ and $\boldsymbol{X}$ and if g^M_m and g^m_M are the *shifters** that effect finite parallel displacement, then the vector connecting $\boldsymbol{P}$ to $\boldsymbol{p}$ is given by

$$g^K_k\,p^k - P^K \quad\text{or}\quad p^k - g^k_K\,P^K,$$

according as the co-ordinates at $\boldsymbol{X}$ or at $\boldsymbol{x}$ are employed.

* Let co-ordinates of $\boldsymbol{X}$ and $\boldsymbol{x}$ in a single rectangular Cartesian system be $\boldsymbol{Z}$ and $\boldsymbol{z}$. The shifters are then given by

$$g^M_m = \delta^K_k\,\frac{\partial X^M}{\partial Z^K}\,\frac{\partial z^k}{\partial x^m}, \qquad g^m_M = \delta^k_K\,\frac{\partial x^m}{\partial z^k}\,\frac{\partial Z^K}{\partial X^M}.$$

Alternatively, they may be defined as those double tensors of the indicated variance such that when both systems of co-ordinates are the same rectangular Cartesian system, we have $g^M_m = \delta^M_m$, $g^m_M = \delta^m_M$.

Let both the space of $\boldsymbol{x}$ and the space of $\boldsymbol{X}$ be metric. Covariant differentiation with respect to x^k and X^K may be defined in the usual way, if we adjoin the convention that $\boldsymbol{X}$ *is held constant* when we differentiate with respect to x^k, and *vice versa.* These *partial* covariant derivatives, which we denote by the usual symbols "$_{,k}$" and "$_{,K}$", are double tensor fields of the type indicated by the number and positions of their indices, and the formal rules of ordinary covariant differentiation remain valid.

Now suppose $n \geqq N$ and consider a mapping $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X})$. The gradients $\partial x^k/\partial X^K$ are double tensor fields, which we denote by

$$x^k_{;K} \equiv \frac{\partial x^k}{\partial X^K}. \tag{3.2}$$

The *total covariant derivative* of a double tensor field $T^{\cdots}_{\cdots}$ is defined by

$$T^{\cdots}_{\dots;K} \equiv T^{\cdots}_{\dots,K} + T^{\cdots}_{\dots,k}\, x^k_{;K}. \tag{3.3}$$

For a given $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{X})$, the total covariant derivative $T^{\cdots}_{\dots;K}$ has a value independent of whether or not $\boldsymbol{x}$ be eliminated through the functional relation $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X})$ in some or all of its occurrences in $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{X})$. This property is not shared by the partial covariant derivative $T^{\cdots}_{\dots,K}$. These remarks are most easily illustrated for the case when $T^{\cdots}_{\cdots}$ reduces to a scalar function $T(x, X)$ of two real variables x and X; the partial covariant derivatives then reduce to $\partial T/\partial x$ and $\partial T/\partial X$, while the total covariant derivative reduces to $\partial T/\partial X + (\partial T/\partial x)\,(dx/dX)$.

The formalism just introduced we shall use in two senses. In the second, $n=3$ and $N=1$ or 2, and the space of $\boldsymbol{X}$ is a curve or surface imbedded in Euclidean three-dimensional space. In the first, $n=N=3$, and we regard $\boldsymbol{X}$ and $\boldsymbol{x}$ the co-ordinates of the same point before and after deformation of the material; in the now usual terminology of the subject, $\boldsymbol{X}$ is a *material co-ordinate* and $\boldsymbol{x}$ a *spatial co-ordinate.* The relation $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X})$ is assumed uniquely invertible to yield $\boldsymbol{X}=\boldsymbol{X}(\boldsymbol{x})$, and we define

$$X^K_{;k} \equiv \frac{\partial X^K}{\partial x^k}, \tag{3.4}$$

$$T^{\cdots}_{\dots;k} \equiv T^{\cdots}_{\dots,k} + T^{\cdots}_{\dots,K}\, X^K_{;k}. \tag{3.5}$$

We have also the chain rule:

$$T^{\cdots}_{\dots;K} = T^{\cdots}_{\dots;k}\, x^k_{;K}, \qquad T^{\cdots}_{\dots;k} = T^{\cdots}_{\dots;K}\, X^K_{;k}. \tag{3.6}$$

In this memoir all considerations are local, and the functions occurring are assumed to have as many continuous derivatives as needed to justify the formal operations.

4. Formulae from the general theory of strain of position. We consider two metric spaces of the same dimension n, with points $\boldsymbol{X}$ and $\boldsymbol{x}$ and with positive definite squared elements of arc:

$$dS^2 = g_{KM}\, dX^K\, dX^M, \qquad ds^2 = g_{km}\, dx^k\, dx^m. \tag{4.1}$$

Under a deformation $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X})$, $\boldsymbol{X}=\boldsymbol{X}(\boldsymbol{x})$, we have

$$\begin{aligned} dS^2 &= c_{km}\, dx^k\, dx^m, & ds^2 &= C_{KM}\, dX^K\, dX^M, \\ c_{km} &= g_{KM}\, X^K_{;k}\, X^M_{;m}, & C_{KM} &= g_{km}\, x^k_{;K}\, x^m_{;M}. \end{aligned} \tag{4.2}$$

The tensors $\boldsymbol{c}$ and $\boldsymbol{C}$ are the *deformation tensors*. Since they are positive definite, they have unique inverses and square roots, which we denote by $\overset{-1}{\boldsymbol{C}}$, $\overset{\frac{1}{2}}{\boldsymbol{C}}$, *etc.* The principal axes of $\boldsymbol{c}$ and $\boldsymbol{C}$ are called the *principal axes of strain* at $\boldsymbol{x}$ and $\boldsymbol{X}$.

A non-singular real matrix $\boldsymbol{a}$ has the unique decompositions $\boldsymbol{a}=\boldsymbol{s}\cdot\boldsymbol{o}=\boldsymbol{o}\cdot\boldsymbol{s}^*$, where $\boldsymbol{o}$ is an orthogonal matrix and where $\boldsymbol{s}$ and $\boldsymbol{s}^*$ are symmetric and positive definite matrices. In a Euclidean space, application of this *polar decomposition theorem* shows that

$$x^k_{;K} = R^k_{\ M}\, \overset{\frac{1}{2}}{C}{}^M_K = \overset{-\frac{1}{2}}{c}{}^k_m\, R^m_{\ K}, \tag{4.3}$$

where $\boldsymbol{C}$ and $\boldsymbol{c}$ are defined above and where $\boldsymbol{R}$ is a rotation tensor*. This is a formal expression of the theorem that any local deformation may be resolved uniquely into a pure strain followed or preceded by a rotation. The rotation carries the principal axes of strain at $\boldsymbol{X}$ into the principal axes of strain at $\boldsymbol{x}$.

Part II. Theory of strain

A. General theory of strain of orientation

5. *Differential description of the unstrained body.* To the point $\boldsymbol{X}$ assign a set of p vectors $\boldsymbol{D}_{\mathfrak{a}}(\boldsymbol{X})$, $\mathfrak{a}=1, 2, \dots, p$, the *directors* of the body at $\boldsymbol{X}$. By a *deformation* we shall mean a transformation carrying $\boldsymbol{X}$ into $\boldsymbol{x}$ and the directors $\boldsymbol{D}_{\mathfrak{a}}$ at $\boldsymbol{X}$ into directors $\boldsymbol{d}_{\mathfrak{a}}$ at $\boldsymbol{x}$. In equations,

$$\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X}), \quad \boldsymbol{d}_{\mathfrak{a}}=\boldsymbol{d}_{\mathfrak{a}}(\boldsymbol{D}_{\mathfrak{b}})=\boldsymbol{d}_{\mathfrak{a}}(\boldsymbol{X}); \tag{5.1}$$

in geometrical terms, a deformation consists in a displacement of the points and independent rotations and stretches of the directors. In the special case when $d^k_{\mathfrak{a}}=x^k_{;K} D^K_{\mathfrak{a}}$, the directors are material elements, and their presence adds nothing to the usual description of strain**. In the special case when $d^k_{\mathfrak{a}}=g^k_K D^K_{\mathfrak{a}}$, the

* An *orthogonal tensor* $\boldsymbol{o}$ in a metric space is characterized by the property $g_{km}\, o^k_{\ p} o^m_{\ q} = g_{pq}$. When $g_{km}=\delta_{km}$, the matrix $\|o^k_{\ p}\|$ is an orthogonal matrix, but in general co-ordinates it is not. In a Euclidean space we may displace fields from one point to another by means of the shifters g^k_K and g^K_k, as mentioned in § 3. A *rotation tensor* such as that occurring in (4.3) is a shifted proper orthogonal tensor: $R^k_{\ M}=g^m_M\, o^k_{\ m}$, $\det o^k_{\ m}=1$.

** An exception should be made for the theory of anisotropic solids, where the directors $\boldsymbol{D}_{\mathfrak{a}}$ are material elements selected on the basis of *a priori* knowledge concerning the nature of the undeformed body. The symmetries of the material are stated in terms of invariance with respect to certain transformation of the $\boldsymbol{D}_{\mathfrak{a}}$. Since $d^k_{\mathfrak{a}}=x^k_{;K}\, D^K_{\mathfrak{a}}$, from (4.2) we obtain

$$g_{\mathfrak{a}\mathfrak{b}} = C_{KM}\, D^K_{\mathfrak{a}}\, D^M_{\mathfrak{b}}, \quad C_{KM} = g_{\mathfrak{a}\mathfrak{b}}\, D^{\mathfrak{a}}_K\, D^{\mathfrak{b}}_M, \quad \overset{-1}{c}{}^{km} = G^{\mathfrak{a}\mathfrak{b}}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}},$$

etc. A formalism of this kind for the strain of anisotropic bodies has been constructed by Ericksen & Rivlin [1954, § 2].

directors are invariable elements and again add nothing*. In general, the directors are *neither material nor invariable*; *i.e.*

$$d_{\mathfrak{a}}^{k} \neq x^{k}_{;K} D_{\mathfrak{a}}^{K} \quad \text{and} \quad d_{\mathfrak{a}}^{k} \neq g^{k}_{K} D_{\mathfrak{a}}^{K}.$$

In an oriented body, strain and rotation are defined from (4.2) and (4.3) in the classical way. What is new is the relation between the directors. For the full range of interpretation mentioned at the end of § 1, it should be possible to use an arbitrarily large number of directors. This degree of generality, however, we do not attempt, being content to consider in a space of n dimensions a set of n linearly independent directors $\boldsymbol{D}_{\mathfrak{a}}$. This number suffices for theories of rods and shells, which we regard as curves and surfaces embedded in a Euclidean space of three dimensions, the maximum number of linearly independent directors thus being, in both cases, three.

Let $\boldsymbol{D}^{\mathfrak{b}}$ be the reciprocals to the $\boldsymbol{D}_{\mathfrak{a}}$, so that

$$D_{\mathfrak{a}}^{K} D_{M}^{\mathfrak{a}} = \delta_{M}^{K}, \quad D_{\mathfrak{a}}^{K} D_{K}^{\mathfrak{b}} = \delta_{\mathfrak{b}}^{\mathfrak{a}}, \quad D_{\mathfrak{a}}^{K} = g^{KM} G_{\mathfrak{a}\mathfrak{b}} D_{M}^{\mathfrak{b}}, \tag{5.2}$$

where** $G_{\mathfrak{a}\mathfrak{b}} \equiv D_{\mathfrak{a}}^{K} D_{\mathfrak{b}}^{M} g_{KM}$. The $\boldsymbol{D}^{\mathfrak{a}}$ are the *reciprocal directors*.

The director triads may be used to define anholonomic components. For example, if we set

$$X_{k}^{\mathfrak{a}} \equiv D_{K}^{\mathfrak{a}} X^{K}_{;k}, \tag{5.3}$$

then $X^{K}_{;k} = D_{\mathfrak{a}}^{K} X_{k}^{\mathfrak{a}}$, and from $(4.2)_3$ follows

$$c_{km} = G_{\mathfrak{a}\mathfrak{b}} X_{k}^{\mathfrak{a}} X_{m}^{\mathfrak{b}}, \quad \textit{etc.} \tag{5.4}$$

Now set

$$W^{K}{}_{MP} \equiv D^{K}_{\mathfrak{a};P} D_{M}^{\mathfrak{a}} = -D_{\mathfrak{a}}^{K} D^{\mathfrak{a}}_{M;P}, \tag{5.5}$$

where for uniformity with later developments we use the notation (3.6) of the total covariant derivative, although of course $D^{K}_{\mathfrak{a};P} = D^{K}_{\mathfrak{a},P}$. From (5.5) follows

$$D^{K}_{\mathfrak{a};M} = W^{K}{}_{PM} D_{\mathfrak{a}}^{P}. \tag{5.6}$$

From (5.5) and (5.2) we find that

$$2\,W_{(KM)P} = G_{\mathfrak{a}\mathfrak{b};P} D_{K}^{\mathfrak{a}} D_{M}^{\mathfrak{b}}. \tag{5.7}$$

Therefore $G_{\mathfrak{a}\mathfrak{b}} = \text{const.}$ is equivalent to

$$W_{KMP} = -W_{MKP}. \tag{5.8}$$

---

* Selecting three orthogonal and invariable directors, we may use them as a fixed anholonomic frame and so obtain invariant forms for the results given in many of the older treatments of finite strain, where all quantities are referred to a common rectangular Cartesian co-ordinate system.

** Note that

$$\det G_{\mathfrak{a}\mathfrak{b}} = \det(D_{\mathfrak{a}} D_{\mathfrak{b}} \cos\vartheta_{\mathfrak{a}\mathfrak{b}}) = (D_1 D_2 \dots D_n)^2 \det\cos\vartheta_{\mathfrak{a}\mathfrak{b}},$$

where $D_{\mathfrak{a}}$ is the length of $\boldsymbol{D}_{\mathfrak{a}}$ and $\vartheta_{\mathfrak{a}\mathfrak{b}}$ is the angle between $\boldsymbol{D}_{\mathfrak{a}}$ and $\boldsymbol{D}_{\mathfrak{b}}$ .When the directors form an orthogonal unit set we thus obtain $\det G_{\mathfrak{a}\mathfrak{b}} = 1$.

In this case, also, if we transform the $\boldsymbol{D}_{\mathfrak{a}}$ at all points by the same orthogonal transformation, the components $W^K{}_{MP}$ are invariant. From these results it follows that if the length of the directors and the angles between them are fixed, as is the case, for example, if the directors are chosen as an orthogonal unit triad, and if the point $\boldsymbol{X}$ is made to traverse the X^P co-ordinate curve at unit speed, the quantities W_{KMP} are the components of angular velocity of the director frame $\boldsymbol{D}_{\mathfrak{a}}$ carried by $\boldsymbol{X}$. We do not need to use (5.8), and therefore we do not impose the restriction $G_{\mathfrak{a}\mathfrak{b}} = \text{const.}$, but this special case serves to motivate our calling $W^K{}_{MP}$ the *wryness* of the director frame in the undeformed material.

6. Differential description of deformation. Results dual to those in the previous section, obtained by systematic interchange of majuscule and minuscule letters, hold also for the deformed material, but the dual wryness tensor, since it refers only to the relative configurations of the director frames at different points in the deformed material, does not afford a comparison between the deformed and undeformed conditions. What we wish, in the kinematic terms used above, are generalizations of the angular velocities of the director frame at $\boldsymbol{x}$ relative to those of the director frame at $\boldsymbol{X}$ when $\boldsymbol{x}$ traverses the curve into which the path of $\boldsymbol{X}$ is deformed. To this end, introduce the *relative wryness* at $\boldsymbol{x}$:

$$F^k{}_{mP} \equiv d^k_{\mathfrak{a};P}\, d^{\mathfrak{a}}_m = -\, d^k_{\mathfrak{a}}\, d^{\mathfrak{a}}_{m;P}, \qquad d^k_{\mathfrak{a};K} = F^k{}_{mK}\, d^m_{\mathfrak{a}}. \tag{6.1}$$

Here, however, we encounter the quantities $d^k_{\mathfrak{a};K}$, where $d^k_{\mathfrak{a};K}$ is the total covariant derivative defined by (3.3); these *director gradients* appear in the theory of deformation of oriented bodies along with the deformation gradients $x^k_{;K}$ as primary local variables. In general there are 27 director gradients; when the directors from an orthogonal unit triad, only 9 director gradients are independent.

Set

$$A^k{}_K \equiv d^k_{\mathfrak{a}}\, D^{\mathfrak{a}}_K, \qquad a^K{}_k \equiv D^K_{\mathfrak{a}}\, d^{\mathfrak{a}}_k; \tag{6.2}$$

then

$$d^k_{\mathfrak{a}} = A^k{}_K\, D^K_{\mathfrak{a}}, \qquad D^K_{\mathfrak{a}} = a^K{}_k\, d^k_{\mathfrak{a}}, \qquad \boldsymbol{a} = \boldsymbol{A}^{-1}. \tag{6.3}$$

From (6.1) follows

$$F^k{}_{mP} = A^k{}_{K;P}\, a^K{}_m + A^k{}_K\, W^K{}_{MP}\, a^M{}_m. \tag{6.4}$$

A deformation of an oriented body is *rigid* if not only $\boldsymbol{C} = \boldsymbol{1}$ but also the directors at $\boldsymbol{x}$ may be obtained from those at $\boldsymbol{X}$ by a uniform orthogonal transformation. In this case the tensor $\boldsymbol{A}$ defined by (6.2) is a covariantly constant orthogonal tensor, the first term on the right-hand side of (6.4) vanishes, and we see that $\boldsymbol{F}$ is *orthogonally equivalent to* $\boldsymbol{W}$.

To make use of these results, we consider the anholonomic components of $\boldsymbol{W}$ with respect to the directors at $\boldsymbol{X}$, the anholonomic components of $\boldsymbol{F}$ with respect to the directors at $\boldsymbol{x}$:

$$\begin{aligned} W_{\mathfrak{a}\mathfrak{b}P} &\equiv W_{KMP}\, D^K_{\mathfrak{a}}\, D^M_{\mathfrak{b}} = D^K_{\mathfrak{a}}\, D_{\mathfrak{b}K;P}, \\ F_{\mathfrak{a}\mathfrak{b}P} &\equiv F_{kmP}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}} = d^k_{\mathfrak{a}}\, d_{\mathfrak{b}k;P}, \end{aligned} \tag{6.5}$$

so that by (5.6) and (6.1) we have

$$D_{\mathfrak{b}K;M} = W_{\mathfrak{a}\mathfrak{b}M}\, D^{\mathfrak{a}}_K, \qquad d_{\mathfrak{b}k;M} = F_{\mathfrak{a}\mathfrak{b}M}\, d^{\mathfrak{a}}_k. \tag{6.6}$$

If we put $g_{\mathfrak{a}\mathfrak{b}} \equiv g_{km}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}}$, then (6.4) assumes the form*

$$F_{\mathfrak{a}\mathfrak{b}P} = A^k{}_{K;P}\, D^K_{\mathfrak{b}}\, d_{\mathfrak{a}k} + g_{\mathfrak{a}\mathfrak{c}}\, G^{\mathfrak{e}\mathfrak{c}}\, W_{\mathfrak{e}\mathfrak{b}P}. \tag{6.7}$$

Also $2F_{(\mathfrak{a}\mathfrak{b})P} = g_{\mathfrak{a}\mathfrak{b};P}$.

Now $\boldsymbol{A}$ is an orthogonal tensor if and only if $g_{\mathfrak{a}\mathfrak{b}} = G_{\mathfrak{a}\mathfrak{b}}$. When $\boldsymbol{A}$ is a uniform orthogonal tensor, (6.7) reduces to

$$F_{\mathfrak{a}\mathfrak{b}P} = W_{\mathfrak{a}\mathfrak{b}P}. \tag{6.8}$$

Conversely, suppose (6.8) holds. From (6.5) follows

$$g_{km}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b};P} = g_{KM}\, D^K_{\mathfrak{a}}\, D^M_{\mathfrak{b};P}. \tag{6.9}$$

Since this holds for all choices of $\mathfrak{a}$ and $\mathfrak{b}$, we deduce

$$g_{\mathfrak{a}\mathfrak{b};P} = G_{\mathfrak{a}\mathfrak{b};P}. \tag{6.10}$$

Hence

$$g_{\mathfrak{a}\mathfrak{b}} = G_{\mathfrak{a}\mathfrak{b}} + K_{\mathfrak{a}\mathfrak{b}}, \tag{6.11}$$

where the $K_{\mathfrak{a}\mathfrak{b}}$ are constants of integration representing constant differences of length and angle. Thus (6.11) asserts that the lengths and angles of the two sets of directors differ by constants, so that the tensor $\boldsymbol{A}$ is orthogonal everywhere if it is orthogonal at one point. From the foregoing analysis we conclude that *necessary and sufficient conditions for rigid deformation of an oriented body are:*

1. $C_{KM} = g_{KM}$.
2. $F_{\mathfrak{a}\mathfrak{b}K} = W_{\mathfrak{a}\mathfrak{b}K}$. (6.12)
3. *At some one point, $\boldsymbol{A}$ is orthogonal.*

The tensor $\boldsymbol{W}$ thus appears as analogous to the metric tensor $\boldsymbol{g}$, while $\boldsymbol{F}$ is analogous to Green's deformation tensor, $\boldsymbol{C}$. However, we must bear in mind that while $(6.12)_1$ is a general tensorial condition, $(6.12)_2$ is not, since the anholonomic components $\boldsymbol{F}$ and $\boldsymbol{W}$ are calculated with respect to different frames. *Cf.* also the footnote to this page.

The relative wryness $\boldsymbol{F}$ is thus a measure of *strain of orientation*, as contrasted with the *strain of position* described by the classical theory of strain. Just as

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* Since the anholonomic components of $\boldsymbol{W}$ and $\boldsymbol{F}$ are defined with respect to *different* anholonomic frames, the usual rules for manipulating anholonomic components do not always apply. For our purposes the particular choices (6.5) for the anholonomic components are essential. For example if instead we set

$$W^{\mathfrak{a}}{}_{\mathfrak{b}P} \equiv W^K{}_{MP}\, D^{\mathfrak{a}}_K\, D^M_{\mathfrak{b}}, \qquad F^{\mathfrak{a}}{}_{\mathfrak{b}P} \equiv F^k{}_{mP}\, d^{\mathfrak{a}}_k\, d^m_{\mathfrak{b}},$$

we obtain from (6.4)

$$F^{\mathfrak{a}}{}_{\mathfrak{b}P} = A^k{}_{K;P}\, d^{\mathfrak{a}}_k\, D^K_{\mathfrak{b}} + W^{\mathfrak{a}}{}_{\mathfrak{b}P}.$$

Therefore a necessary and sufficient condition for $\boldsymbol{A}$ to be covariantly constant is

$$F^{\mathfrak{a}}{}_{\mathfrak{b}P} = W^{\mathfrak{a}}{}_{\mathfrak{b}P},$$

but this is not at all the same as (6.8).

the C_{MP} are certain quadratic combinations of the deformation gradients $x^k_{;K}$, the $F_{\mathfrak{a}\mathfrak{b}M}$ are certain linear combinations of the director gradients $d^{\mathfrak{c}}_{k;K}$. In general, the numbers of independent quantities $x^k_{;K}$, C_{KM}, $d^{\mathfrak{a}}_{k;K}$, and $F_{\mathfrak{a}\mathfrak{b}M}$ are, respectively, 9, 6, 27, 27.

Just as there are many alternative and equally correct measures of strain of position, so are there also other possible measures of strain of orientation besides $F_{\mathfrak{a}\mathfrak{b}M}$, but we do not take up the question of characterizing the class of strain measures.

By systematic interchange of majuscules and minuscules we may obtain a dual description in which the deformed rather than the undeformed body is the standard of reference.

Not pausing to explore the structure whose traits have just been presented, we note only a formula for the gradient of the relative wryness. By differentiating (6.1) we get

$$\begin{aligned} F^k{}_{mM;K} &= d^k_{\mathfrak{a};MK}\, d^{\mathfrak{a}}_m + d^k_{\mathfrak{a};M}\, d^{\mathfrak{a}}_{m;K}, \\ &= d^k_{\mathfrak{a};MK}\, d^{\mathfrak{a}}_m - F^k{}_{pM}\, F^p{}_{mK}. \end{aligned} \tag{6.13}$$

Hence

$$\begin{aligned} F_{\mathfrak{a}\mathfrak{b}M;K} &= F_{kmM;K}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}} + F_{kmM}\,(d^k_{\mathfrak{a};K}\, d^m_{\mathfrak{b}} + d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b};K}), \\ &= d_{\mathfrak{b}k;MK}\, d^k_{\mathfrak{a}} + g^{\mathfrak{c}\mathfrak{e}}(F_{\mathfrak{a}\mathfrak{e}M}\, F_{\mathfrak{c}\mathfrak{b}K} + F_{\mathfrak{e}\mathfrak{b}M}\, F_{\mathfrak{c}\mathfrak{a}K} - F_{\mathfrak{a}\mathfrak{c}M}\, F_{\mathfrak{e}\mathfrak{b}K}). \end{aligned} \tag{6.14}$$

The foregoing analysis is freely adapted from that of E. & F. COSSERAT [1909, §§ 48–50].

B. Rods

7. Strain of position. Let a curve $\mathscr{C}$ in Euclidean three-dimensional space be given by the parametric equations

$$X^K = X^K(S), \qquad K = 1, 2, 3, \tag{7.1}$$

where it is convenient to regard S as arc length. In a deformation $\mathscr{C}$ is mapped onto a curve c given by

$$x^k = x^k(s), \qquad k = 1, 2, 3, \tag{7.2}$$

where

$$s = s(S). \tag{7.3}$$

The *stretch* λ is given by

$$\lambda = \frac{ds}{dS}. \tag{7.4}$$

A necessary and sufficient condition that corresponding sections of c and $\mathscr{C}$ have the same lengths is $\lambda = 1$. This completes the intrinsic theory of strain of position.

For the imbedding theory of strain of position, we remind the reader that if $\mathscr{C}$ and c have the same curvature and torsion as functions of S, then one may be brought into point by point coincidence with the other by means of suitable rigid motion. Thus a complete description of the strain of position, as far as the imbedding theory is concerned, is given by the curvatures and torsions of c and $\mathscr{C}$ as functions of S.

8. Alternative interpretation. The foregoing approach is strictly one-dimensional, but for some purposes of interpretation it is preferable to consider the curve $\mathscr{C}$ not as isolated but rather as imbedded in a three-dimensional material and deformed with it so as to assume the configuration c. This view is legitimate in all cases, since, given (7.1) and (7.2) and (7.3), we may always imagine many deformations of a three-dimensional body such as to carry the particular curve $\mathscr{C}$ into the particular curve c. From the three-dimensional theory of strain, we may calculate the stretch λ of the unit tangent $\boldsymbol{T}$ to $\mathscr{C}$, obtaining

$$\lambda^2 = C_{KM}\, T^K\, T^M, \tag{8.1}$$

where $\boldsymbol{C}$ is defined by (4.2), and this must agree with (7.4). In fact, a necessary and sufficient condition to be satisfied by the three-dimensional deformation is that one point of $\mathscr{C}$ be carried into the corresponding point of c and that (8.1) be compatible with (7.4).

In what follows, we shall shift from one interpretation to the other as seems helpful. For example, for a three-dimensional double field $\boldsymbol{A}(\boldsymbol{X}, \boldsymbol{x})$, evaluated at points on $\mathscr{C}$ and c, we have $\boldsymbol{A} = \boldsymbol{A}\left(\boldsymbol{X}(S), \boldsymbol{x}(s)\right)$, and we may set

$$\hat{A}^{\cdots}_{\cdots} \equiv A^{\cdots}_{\cdots;K} \frac{dX^K}{dS}. \tag{8.2}$$

From the definition (3.3) follows

$$\hat{A}^{\cdots}_{\cdots} = \frac{\partial A^{\cdots}_{\cdots}}{\partial S} + \frac{\partial A^{\cdots}_{\cdots}}{\partial s}\frac{ds}{dS} + \{\,\dot{}_{K}\}\, A^{\cdots}_{\cdots}\frac{dX^K}{dS} + \cdots + \{\,\dot{}_{k}\}\, A^{\cdots}_{\cdots}\frac{dx^k}{ds}\frac{ds}{dS}, \tag{8.3}$$

and in this formula appear only quantities which can be calculated from the equations of the curves $\mathscr{C}$ and c and the deformation $s = s(S)$. For a strictly one-dimensional approach, (8.3) rather than the simpler and more understandable formula (8.2) should be taken as the *definition* of $\hat{\boldsymbol{A}}$; as far as results are concerned, it makes no difference.

In conformity with this notation, set

$$\hat{X}^K \equiv \frac{dX^K}{dS}, \qquad \hat{x}^k \equiv \frac{dx^k}{ds}\frac{ds}{dS}, \tag{8.4}$$

where X^K and x^k are given parametrically by (7.1) and (7.2). Note also that for a scalar $F(S)$ we have simply

$$\hat{F} = \frac{dF}{dS}. \tag{8.5}$$

9. Differential description of the undeformed rod. A *rod* is a curve $\mathscr{C}$ at each point of which are assigned three linearly independent directors $\boldsymbol{D}_{\mathfrak{a}}$. We set

$$T^{\mathfrak{a}} \equiv D^{\mathfrak{a}}_K \hat{X}^K, \qquad G_{\mathfrak{a}\mathfrak{b}} \equiv g_{KM} D^K_{\mathfrak{a}} D^M_{\mathfrak{b}}, \qquad W^{\mathfrak{a}}{}_{\mathfrak{b}} \equiv D^{\mathfrak{a}}_K \hat{D}^K_{\mathfrak{b}}, \tag{9.1}$$

so that

$$\hat{X}^K = D^K_{\mathfrak{a}} T^{\mathfrak{a}}, \quad \hat{D}^K_{\mathfrak{a}} = D^K_{\mathfrak{b}} W^{\mathfrak{b}}{}_{\mathfrak{a}}, \quad 2W_{(\mathfrak{a}\mathfrak{b})} = \hat{G}_{\mathfrak{a}\mathfrak{b}}, \quad G_{\mathfrak{a}\mathfrak{b}} T^{\mathfrak{a}} T^{\mathfrak{b}} = 1. \tag{9.2}$$

The scalars $T^{\mathfrak{a}}$, anholonomic components of the unit tangent, are the lengths of the projections of the unit tangent upon the reciprocal directors; the $G_{\mathfrak{a}\mathfrak{b}}$ prescribe the lengths and mutual angles of the directors; the tensor $\boldsymbol{W}$, whose

anholonomic components are the rates of change of the directors resolved along the directions of the reciprocal directors, is the *wryness* of the directors along $\mathscr{C}$. By (9.2) we see that if the $G_{\mathfrak{a}\mathfrak{b}}$ are constant along $\mathscr{C}$, and in that case only, $\boldsymbol{W}$ may be regarded as an angular velocity, as explained following (5.8). Sometimes it is preferable to use tensor components rather than anholonomic components:

$$W^K{}_M = \hat{D}^K_{\mathfrak{a}} D^{\mathfrak{a}}_M = D^K_{\mathfrak{b}} W^{\mathfrak{b}}{}_{\mathfrak{a}} D^{\mathfrak{a}}_M, \qquad \hat{D}^K_{\mathfrak{a}} = W^K{}_M D^M_{\mathfrak{a}}, \qquad W^{\mathfrak{a}}{}_{\mathfrak{b}} = D^{\mathfrak{a}}_K W^K{}_M D^M_{\mathfrak{b}}. \tag{9.3}$$

(If $\mathscr{C}$ is thought of as imbedded in a three-dimensional oriented body, the wryness $W^{\mathfrak{a}}{}_{\mathfrak{b}}$ along $\mathscr{C}$ is related to the wryness of the body, *viz.* $W^K{}_{MP}$, as follows:

$$W^{\mathfrak{a}}{}_{\mathfrak{b}} = D^{\mathfrak{a}}_K W^K{}_M D^M_{\mathfrak{b}}, \qquad W^K{}_M = W^K{}_{MP} \hat{X}^P.) \tag{9.4}$$

Suppose the quantities $T^{\mathfrak{a}}$, $W^{\mathfrak{a}}{}_{\mathfrak{b}}$, and $G_{\mathfrak{a}\mathfrak{b}} = G_{\mathfrak{b}\mathfrak{a}}$ be given as any functions of S such that $(9.2)_3$ and $(9.2)_4$ are satisfied and such that $\boldsymbol{G}$ is positive definite. Equations $(9.2)_{1,2}$ may then be regarded as a differential system for determining $X^K(S)$ and $D^K_{\mathfrak{a}}(S)$. From (9.2), any solution satisfies

$$\overline{G_{KM} D^K_{\mathfrak{a}} D^M_{\mathfrak{b}}}^{\wedge} = 2 G_{KM} D^K_{\mathfrak{c}} D^M_{(\mathfrak{a}} W^{\mathfrak{c}}{}_{\mathfrak{b})}. \tag{9.5}$$

We may write $(9.2)_3$ in the form $\hat{G}_{\mathfrak{a}\mathfrak{b}} = 2 G_{\mathfrak{c}(\mathfrak{a}} W^{\mathfrak{c}}{}_{\mathfrak{b})}$; comparing this result with (9.5) shows that $G_{KM} D^K_{\mathfrak{a}} D^M_{\mathfrak{b}}$ and $G_{\mathfrak{a}\mathfrak{b}}$ satisfy the same first order differential system. By the uniqueness of solutions of such systems, $(9.1)_2$ will hold for all S if it holds for one value of S. Given a solution $X^K(S)$, $D^K_{\mathfrak{a}}(S)$ of the system $(9.2)_{1,2}$, suppose $(9.1)_2$ is satisfied; by $(9.2)_4$ follows

$$\hat{X}^K \hat{X}_K = G_{\mathfrak{a}\mathfrak{b}} T^{\mathfrak{a}} T^{\mathfrak{b}} = 1, \tag{9.6}$$

so the solution represents an oriented rod with S as arc length. It is easily shown that two sets of initial data consistent with $(9.1)_2$ are related by a rigid motion, perhaps combined with a reflection, and that any rigid motion combined with a reflection carries any solution of the system $(9.2)_{1,2}$ into another. Uniqueness of solutions of this system thus implies that *assignment* of $T^{\mathfrak{a}}$, $W^{\mathfrak{a}}{}_{\mathfrak{b}}$ *and a positive definite* $G_{\mathfrak{a}\mathfrak{b}}$ *satisfying the conditions* $(9.2)_{3,4}$ *determines* $\mathscr{C}$ *and its directors to within a rigid motion combined with a reflection.*

10. Special cases: curvature, torsion, and twist. In the special case when the directors are chosen as the unit tangent, principal normal and binormal to $\mathscr{C}$, oriented so as to form a right-handed system, the Serret-Frenet formulae assert that

$$\|W^{\mathfrak{a}}{}_{\mathfrak{b}}\| = \left\| \begin{matrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{matrix} \right\|, \tag{10.1}$$

where $\varkappa$ and τ are the curvature and the torsion of $\mathscr{C}$. No such result holds for a general triad of directors or would be of interest here, since we seek not the properties of the curve itself but rather of a set of axes attached to the curve but turning independently of it.

For another special case, consider a single unit director $\boldsymbol{D}$ which is normal to the curve $\mathscr{C}$, and let it subtend an angle $\varphi(S)$ with the principal normal $\boldsymbol{N}$,

measured positively toward the binormal, so that

$$D^K = N^K \cos\varphi + B^K \sin\varphi \tag{10.2}$$

where $\boldsymbol{B}$ is the binormal. The quantity $\hat{\varphi}$ is the *twist* of $\boldsymbol{D}$ with respect to $\mathscr{C}$. If both the curve $\mathscr{C}$ and the director $\boldsymbol{D}$ are subjected to the same orthogonal transformation, $\hat{\varphi}$ is unchanged. Since $D_K N^K = \cos\varphi$, we have

$$\hat{D}^K N_K + D_K \hat{N}^K = -\sin\varphi\,\hat{\varphi}; \tag{10.3}$$

by $(9.3)_3$ and the Serret-Frenet formula for $\hat{N}$ follows

$$W^K{}_M D^M N_K + D_K(-\varkappa T^K + \tau B^K) = -\sin\varphi\,\hat{\varphi}. \tag{10.4}$$

Substitution of (10.2) in this result yields

$$W_{(KM)} N^K N^M \cos\varphi + (W_{KM} N^K B^M + \tau + \hat{\varphi}) \sin\varphi = 0. \tag{10.5}$$

Thus far in this paragraph we have spoken of but a single director $\boldsymbol{D}$, although for definition of the wryness $\boldsymbol{W}$ by $(9.1)_3$ a set of three directors is required. Since we are here interested only in the single director $\boldsymbol{D}$, and since this is a unit vector, there is no loss of generality in selecting the other two directors as of fixed length and subtending fixed angles with $\boldsymbol{D}$ and with each other. By $(9.2)_3$ follows $W_{(KM)} = 0$, and thus, provided $\varphi \neq 0$, from (10.5) we obtain

$$\hat{\varphi} = W_{MK} N^K B^M - \tau. \tag{10.6}$$

That is, *the twist is the excess of $W_{MK} N^K B^M$ over the torsion*. The scalar $W_{MK} N^K B^M$ is itself an anholonomic component of $\boldsymbol{W}$ with respect to the principal frame of $\mathscr{C}$. If we let $\boldsymbol{D}^*$ be a unit vector such that $\boldsymbol{D}^*$, $\boldsymbol{D}$, and $\boldsymbol{T}$ form a right-handed unit triad, since $D^{*K} = -N^K \sin\varphi + B^K \cos\varphi$ it follows that $W_{KM} B^K N^M = D^{*K} W_{KM} D^M$, so that by $(9.3)_3$ and (10.6) follows

$$\hat{\varphi} = D^*_K \hat{D}^K - \tau. \tag{10.7}$$

This formula is the basis of the classical attempts to construct a theory of strain for bent and twisted rods.

11. Criticism of the classical description of strain of a rod. Since the rotation of $\boldsymbol{D}$ may be prescribed in any smooth way as we traverse $\mathscr{C}$, there can be no general connection between the twists of two different directors. In the applications of the theory to elasticity, it is customary to think of a rod as a line furnished with normal *cross-sections*; these are represented, for the purposes of the theory, only by their principal axes of geometrical inertia and perhaps one or two constants such as their geometrical moments of inertia about these axes. The twist of the unstrained rod is then defined as the twist of either of these axes relative to $\mathscr{C}$. Since these axes are normal to one another, a unique twist is obtained.

Now consider the deformation of $\mathscr{C}$ into c, defined not only by (7.1)—(7.3) but also by a relation setting directors $\boldsymbol{d}_\alpha$ along c into correspondence with the directors $\boldsymbol{D}_\alpha$ along $\mathscr{C}$:

$$\boldsymbol{d}_\alpha = \boldsymbol{d}_\alpha(s, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3). \tag{11.1}$$

Pursuing the interpretation mentioned above, we might choose for one director $\boldsymbol{d}$ a unit vector along the direction into which one of the directors $\boldsymbol{D}$, considered

as material, is deformed according to the three-dimensional theory. Such a director, $\boldsymbol{d}_0$, in general would not be orthogonal to c. In practice, $\boldsymbol{d}$ is selected as the normal to the unit tangent $\boldsymbol{t}$ which lies in the plane of $\boldsymbol{d}_0$ and $\boldsymbol{t}$. The triply orthogonal directions so determined are called the *principal torsion-flexure axes* in the deformed rod. The twist in the deformed rod is then defined as the twist of $\boldsymbol{d}$ relative to c.

Obviously this classical procedure is motivated only by formal simplicity and furnishes an inadequate description of the strain of a rod. In the first place, twist is defined unsymmetrically with respect to the unstrained and strained rods. While the twist of the unstrained rod is uniquely determined, two different twists can be obtained for the strained rod, depending on which of the principal axes of inertia of the cross-section is selected for $\boldsymbol{D}$ in the unstrained rod, and by the above remarks, these two twists are in general entirely unrelated to one another. Moreover, the insistence that $\boldsymbol{d}$ be normal to c is merely artificial and does not represent any kinematical requirement.

12. Exact differential description of the strain of a rod. For a more precise description, we need only adapt to c what has already been done for $\mathscr{C}$. Since the operation "$\hat{\ }$" defined by (8.3) is a derivative with respect to S, not s, the analysis, while parallel, is not merely dual to that given above. We set

$$t^{\mathfrak{a}} \equiv d^{\mathfrak{a}}_k \hat{x}^k, \quad C_{\mathfrak{a}\mathfrak{b}} \equiv g_{km}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}}, \qquad F^{\mathfrak{a}}{}_{\mathfrak{b}} \equiv d^{\mathfrak{a}}_k \hat{d}^k_{\mathfrak{b}}. \tag{12.1}$$

The $C_{\mathfrak{a}\mathfrak{b}}$ are the *components of deformation*; the $F^{\mathfrak{a}}{}_{\mathfrak{b}}$, those of the *relative wryness* of c.

By analogy with what was said earlier in connection with $\mathscr{C}$ it follows that if $t^{\mathfrak{a}}$, $C_{\mathfrak{a}\mathfrak{b}}$, and $F^{\mathfrak{a}}{}_{\mathfrak{b}}$ are given functions of S subject to the conditions that $C_{\mathfrak{a}\mathfrak{b}}$ be symmetric and positive definite and that

$$\hat{C}_{\mathfrak{a}\mathfrak{b}} = 2C_{\mathfrak{c}(\mathfrak{a}} F^{\mathfrak{c}}{}_{\mathfrak{b})}, \tag{12.2}$$

the rod c is determined to within a rigid motion combined with a reflection, its arc length s being obtained by integrating the equation

$$\lambda^2 = \frac{ds^2}{dS^2} = \hat{x}_k \hat{x}^k = C_{\mathfrak{a}\mathfrak{b}}\, t^{\mathfrak{a}} t^{\mathfrak{b}}, \tag{12.3}$$

where λ is the stretch. What has been shown, then, is summarized in the following **fundamental theorem on the strain of a rod**: *Given a rod $\mathscr{C}$ with arc length S and directors $\boldsymbol{D}_{\mathfrak{a}}(S)$, prescription of the 18 scalars $t^{\mathfrak{a}}$, $C_{\mathfrak{a}\mathfrak{b}}$, and $F^{\mathfrak{a}}{}_{\mathfrak{b}}$ as functions of S subject to the aforementioned conditions determines a second rod $\mathscr{C}$ with arc length s and directors $\boldsymbol{d}_{\mathfrak{a}}(s)$, uniquely to within a rigid motion combined with a reflection.* In other words, the quantities $t^{\mathfrak{a}}$, $C_{\mathfrak{a}\mathfrak{b}}$ and $F^{\mathfrak{a}}{}_{\mathfrak{b}}$ furnish *a complete differential description of the strain of a rod*. It suffices to specify only the 12 scalars $t^{\mathfrak{a}}$, $C_{\mathfrak{a}\mathfrak{b}}$, and $C_{\mathfrak{a}[\mathfrak{b}} F^{\mathfrak{a}}{}_{\mathfrak{c}]}$, since one can then calculate $F^{\mathfrak{a}}{}_{\mathfrak{b}}$ by using (12.2).

The apparatus constructed is very general, enough so to include what would be regarded physically as an anisotropic rod. To represent a physically isotropic rod, let $\boldsymbol{D}_1$ be the unit tangent to $\mathscr{C}$ and $\boldsymbol{d}_1$ the unit tangent to c. In this case, $T^1 = t^1 = 1$, $T^2 = T^3 = t^2 = t^3 = 0$. The remaining two directors, both along $\mathscr{C}$ and along c, may be assigned arbitrarily.

13. Rotation. Again we consider a general director frame. To determine the finite rotation of a rod, it is necessary to regard the rod from the Euclidean space in which it is imbedded. So as to motivate the definition shortly to be given, we first consider $\mathscr{C}$ to be a material curve in a three-dimensional body and the directors $\boldsymbol{D}_{\mathfrak{a}}$ *material elements*, this being the case that one-dimensional theories of rods are intended to idealize. Then, as remarked in § 5, in (6.3) we have $A^k{}_K = x^k{}_{;K}$, $a^K{}_k = X^K{}_{;k}$. By the classical theory mentioned in § 4, we can decompose $x^k{}_{;K}$ uniquely into a stretch and a rotation, and the rotation so determined is the rotation of the principal axes of strain at $\boldsymbol{X}$ into the principal axes of strain at $\boldsymbol{x}$. In a purely one-dimensional theory, we do not have available the $x^k{}_{;K}$, but from the two sets of directors we can define $A^k{}_K$ by (6.2), and, since the linear independence of the directors assures us that $A^k{}_K$ is non-singular, from the polar decomposition theorem (§ 4) we can write

$$A^k{}_K = R^k{}_M P^M_K = p^k_m R^m{}_K, \tag{13.1}$$

where $\boldsymbol{R}$ is a rotation tensor and $\boldsymbol{P}$ and $\boldsymbol{p}$ are positive definite symmetric tensors. The tensor $\boldsymbol{R}$ then represents the *local rotation* of the rod. What we have proved regarding it is summarized as follows: *Given two rods with directors $\boldsymbol{D}_{\mathfrak{a}}$ and $\boldsymbol{d}_{\mathfrak{a}}$, a unique local rotation is defined by* (13.1); *the rotation is independent of the choice of the directors to this extent, that if we imagine and then leave fixed a deformation of a three-dimensional body in which $\mathscr{C}$ is a material line and the given directors $\boldsymbol{D}_{\mathfrak{a}}$ are carried materially into the given directors $\boldsymbol{d}_{\mathfrak{a}}$, then we may choose as directors any other set of linearly independent material vectors and obtain the same rotation.*

When the directors are chosen as above, $(12.1)_2$ becomes

$$C_{\mathfrak{a}\mathfrak{b}} = C_{KM}\, D^K_{\mathfrak{a}}\, D^M_{\mathfrak{b}};. \tag{13.2}$$

The quantities $C_{\mathfrak{a}\mathfrak{b}}$ are anholonomic components of the three-dimensional deformation tensor $\boldsymbol{C}$ with respect to the directors of $\mathscr{C}$.

14. Special case: HAY'S *formalism.* The only previous analysis approaching the generality of what has just been presented is that of HAY [1942, §§ 2–3]. In effect, he takes the directors $\boldsymbol{D}_{\mathfrak{a}}$ of the undeformed rod as any unit triad such that $\boldsymbol{D}_1$ is the unit tangent to $\mathscr{C}$, and he chooses co-ordinate systems such that not only $D^K_{\mathfrak{a}} = \delta^K_{\mathfrak{a}}$ but also $d^k_{\mathfrak{a}} = \delta^k_{\mathfrak{a}}$. Hence (6.2) yields $A^k{}_K = \delta^k_K$, $a^K{}_k = \delta^K_k$, but it must be remembered that the co-ordinates at $\boldsymbol{x}$ are not generally orthogonal. In these co-ordinates we have from $(12.1)_2$

$$C_{\mathfrak{a}\mathfrak{b}} = g_{km}\, d^k_{\mathfrak{a}}\, d^m_{\mathfrak{b}} = g_{km}\, \delta^k_{\mathfrak{a}}\, \delta^m_{\mathfrak{b}}; \tag{14.1}$$

thus the components of the metric tensor at $\boldsymbol{x}$ are numerically equal to the components of stretch, this being a generalization of HAY's equation (3.9). From $(12.1)_3$ we get in these co-ordinates

$$\begin{aligned} F^{\mathfrak{a}}_{\mathfrak{b}} &= d^{\mathfrak{a}}_k \left[\frac{\partial d^k_{\mathfrak{b}}}{\partial s} + \left\{ {k \atop m\,p} \right\} d^m_{\mathfrak{b}} \frac{dx^p}{ds}\right] \frac{ds}{dS}, \\ &= \delta^{\mathfrak{a}}_k \left\{ {k \atop m\,1} \right\} \delta^m_{\mathfrak{b}}\, \lambda, \\ &= \delta^{\mathfrak{a}}_k\, \delta^m_{\mathfrak{b}}\, g^{kp} \left(\frac{\partial g_{1[p}}{\partial x^{m]}} + \frac{1}{2}\frac{d g_{pm}}{\partial x^1}\right) \lambda, \end{aligned} \tag{14.2}$$

where the rod c is taken as the x^1-curve with $x^1 = s$. The skew-symmetric part of this equation is essentially HAY's equation (3.5); hence our relative wryness $\boldsymbol{F}$ includes and generalizes HAY's rotation vector $\boldsymbol{\omega}$.

C. Shells

15. Intrinsic theory of strain of position. Let the undeformed surface $\mathscr{S}$ be given by

$$X^K = X^K(V^1, V^2) = X^K(V^\Xi), \tag{15.1}$$

the deformed surface s by

$$x^k = x^k(v^1, v^2) = x^k(v^\xi), \tag{15.2}$$

where, as in the rest of this section, Latin indices run from 1 to 3, Greek indices from 1 to 2. The deformation is specified by the functional forms of the right-hand sides of (15.1) and (15.2), augmented by

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{V}). \tag{15.3}$$

For full generality, it is sufficient but not necessary to take

$$v^\xi = \delta^\xi_\Xi V^\Xi. \tag{15.4}$$

The surface metric tensors are given by

$$\begin{aligned} dS^2 &= A_{\Delta\Xi}\, dV^\Delta\, dV^\Xi = A_{\delta\xi}\, dv^\delta\, dv^\xi, \\ ds^2 &= a_{\delta\xi}\, dv^\delta\, dv^\xi = a_{\Delta\Xi}\, dV^\Delta\, dV^\Xi, \end{aligned} \tag{15.5}$$

where

$$A_{\delta\xi} = A_{\Delta\Xi}\, V^\Delta_{;\delta}\, V^\Xi_{;\xi}, \qquad a_{\Delta\Xi} = a_{\delta\xi}\, v^\delta_{;\Delta}\, v^\xi_{;\Xi}, \tag{15.6}$$

and where ";" denotes a partial derivative. If $\boldsymbol{A}(\boldsymbol{V})$ and $\boldsymbol{a}(\boldsymbol{v})$ are known, and if the parameter transformation (15.3) is given, we may calculate for any material element the changes of length and angle occasioned by the deformation. Also, we may calculate the total curvatures R_{1212} of $\mathscr{S}$ and s. Corresponding to any assigned real symmetric tensor $\boldsymbol{a}(\boldsymbol{v})$, with $\boldsymbol{A}(\boldsymbol{V})$ and the relation (15.3) regarded as given, there exists a surface s into which $\mathscr{S}$ is deformed, and any two such surfaces s_1 and s_2 are applicable. In principle, this completes the intrinsic theory of strain of position of a shell.

In terms of the three-dimensional equations (15.1) and (15.2), we may calculate the induced surface matrices $\boldsymbol{A}$ and $\boldsymbol{a}$, obtaining

$$A_{\Delta\Xi} = g_{KM}\, X^K_{;\Delta}\, X^M_{;\Xi}, \qquad a_{\delta\xi} = g_{km}\, x^k_{;\delta}\, x^m_{;\xi}. \tag{15.7}$$

16. Imbedding theory of strain of position. The second fundamental tensors $\boldsymbol{B}$ and $\boldsymbol{b}$ are given by

$$B_{\Xi\Delta} = N_K\, X^K_{;\Delta\Xi}, \quad b_{\delta\xi} = n_k\, x^k_{;\delta\xi}, \tag{16.1}$$

where $X^M_{;\Delta\Xi} \equiv (X^M_{;\Delta})_{;\Xi}$, $x^m_{;\delta\xi} \equiv (x^m_{;\delta})_{;\xi}$, the total covariant derivative ";" being defined by (3.3), and where $\boldsymbol{N}$ and $\boldsymbol{n}$ are the unit normals to $\mathscr{S}$ and s. We have

$$B_{\Delta\Xi}\, dV^\Delta\, dV^\Xi = B_{\delta\xi}\, dv^\delta\, dv^\xi, \qquad b_{\delta\xi}\, dv^\delta\, dv^\xi = b_{\Delta\Xi}\, dV^\Delta\, dV^\Xi, \tag{16.2}$$

where
$$B_{\delta\xi} = B_{\varDelta\Xi} V^{\varDelta}_{;\delta} V^{\Xi}_{;\xi}, \qquad b_{\varDelta\Xi} = b_{\delta\xi} v^{\delta}_{;\varDelta} v^{\xi}_{;\Xi}. \tag{16.3}$$

The tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ are related by the equations of GAUSS and MAINARDI-CODAZZI:
$$KA = R_{1212} = B, \qquad B_{\varDelta\varDelta;\Xi} - B_{\varDelta\Xi;\varDelta} = 0, \tag{16.4}$$
where K is the Gaussian curvature, $A \equiv \det A_{\varDelta\Xi}$, $B \equiv \det B_{\varDelta\Xi}$. There are dual relations connecting $\boldsymbol{a}$ and $\boldsymbol{b}$. When $\mathscr{S}$ and a parameter mapping (15.3) are given, assignment of arbitrary symmetric tensors $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfying the duals to (16.4) determines a surface s into which $\mathscr{S}$ is deformed, uniquely to within a rigid motion*.

The surface $\mathscr{S}$ is obtained by solving the equations
$$X^{K}_{;\Xi\varDelta} = B_{\Xi\varDelta} N^{K}, \qquad N^{K}_{;\varDelta} = -B^{\Xi}_{\varDelta} X^{K}_{;\Xi}, \tag{16.5}$$
using appropriate assignments of X^K and N^K at one point, and s is obtained by solving the dual equations. In principle, this completes the imbedding theory of strain of position.

The normal curvature $K_{(\boldsymbol{N})}$ of $\mathscr{S}$ in the direction of a unit vector $\boldsymbol{N}$ is calculated from
$$K_{(\boldsymbol{N})} = \frac{B_{\Xi\varDelta} N^{\Xi} N^{\varDelta}}{A_{\varPhi\varPsi} N^{\varPhi} N^{\varPsi}}; \tag{16.6}$$
the principal curvatures K_1 and K_2 are the curvatures $K_{(\boldsymbol{N})}$ in the principal directions of $\boldsymbol{B}$, and the Gaussian curvature $\overline{K}$ and the mean curvature K are related to these principal curvatures through
$$K = K_1 K_2 = \frac{B}{A}, \quad \overline{K} = K_1 + K_2 = A^{\Xi\varDelta} B_{\Xi\varDelta}. \tag{16.7}$$
In approximate theories of elastic shells it is customary to take the changes of principal curvature as measures of strain. To do this in the exact theory would be to introduce complications analogous to those resulting from using the displacement vector and the strain tensors in finite strain of three-dimensional bodies.

17. Differential description of the undeformed shell. A *shell* is a surface $\mathscr{S}$ at each point of which are assigned three linearly independent directors $\boldsymbol{D}_{\mathfrak{a}}$. The theory of strain of orientation for a shell may be constructed in analogy to what was done for three-dimensional bodies in § 5. Set
$$X^{\mathfrak{a}}_{\varDelta} \equiv D^{\mathfrak{a}}_{K} X^{K}_{;\varDelta}, \quad G_{\mathfrak{a}\mathfrak{b}} \equiv G_{KM} D^{K}_{\mathfrak{a}} D^{M}_{\mathfrak{b}}, \quad W^{\mathfrak{a}}{}_{\mathfrak{b}\varDelta} \equiv D^{\mathfrak{a}}_{K} D^{K}_{\mathfrak{b};\varDelta}. \tag{17.1}$$
Then
$$X^{K}_{;\varDelta} = D^{K}_{\mathfrak{a}} X^{\mathfrak{a}}_{\varDelta}, \quad D^{K}_{\mathfrak{a};\varDelta} = D^{K}_{\mathfrak{b}} W^{\mathfrak{b}}{}_{\mathfrak{a}\varDelta}. \tag{17.2}$$
Then by $(15.7)_1$ follows
$$A_{\varDelta\Xi} = G_{\mathfrak{a}\mathfrak{b}} X^{\mathfrak{a}}_{\varDelta} X^{\mathfrak{b}}_{\Xi}, \tag{17.3}$$
while by $(15.8)_1$ follows
$$B_{\varDelta\Xi} \equiv N_K D^{K}_{\mathfrak{a}} (X^{\mathfrak{a}}_{\varDelta;\Xi} + W^{\mathfrak{a}}{}_{\mathfrak{b}\Xi} X^{\mathfrak{b}}_{\varDelta}). \tag{17.4}$$

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* *Cf., e.g.*, EISENHART [1940, § 39].

From the relation (17.2),

$$X^K_{;\Xi\Delta} = D^K_{\mathfrak{b}} (X^{\mathfrak{b}}_{\Xi;\Delta} + W^{\mathfrak{b}}{}_{\mathfrak{a}\Delta} X^{\mathfrak{a}}_{\Xi}), \tag{17.5}$$

$$D^K_{\mathfrak{a};\Xi\Delta} = D^K_{\mathfrak{c}} (W^{\mathfrak{c}}{}_{\mathfrak{b}\Delta} W^{\mathfrak{b}}{}_{\mathfrak{a}\Xi} + W^{\mathfrak{c}}{}_{\mathfrak{a}\Xi;\Delta}). \tag{17.6}$$

We also have the integrability conditions

$$\frac{\partial^2 X^K}{\partial V^{[\Delta}\partial V^{\Xi]}} = X^K_{;[\Delta\Xi]} = 0, \qquad \frac{\partial^2 D^K_{\mathfrak{a}}}{\partial V^{[\Delta}\partial V^{\Xi]}} = D^K_{\mathfrak{a};[\Delta\Xi]} = 0, \tag{17.7}$$

whence follows

$$\frac{\partial X^{\mathfrak{b}}_{[\Delta}}{\partial V^{\Xi]}} + W^{\mathfrak{b}}{}_{\mathfrak{a}[\Xi} X^{\mathfrak{a}}_{\Delta]} = 0, \tag{17.8}$$

$$W^{\mathfrak{c}}{}_{\mathfrak{b}\Xi} W^{\mathfrak{b}}{}_{\mathfrak{a}\Delta} - W^{\mathfrak{c}}{}_{\mathfrak{b}\Delta} W^{\mathfrak{b}}{}_{\mathfrak{a}\Xi} + 2\frac{\partial W^{\mathfrak{c}}{}_{\mathfrak{a}[\Delta}}{\partial V^{\Xi]}} = 0. \tag{17.9}$$

From $(17.1)_2$ and $(17.2)_2$,

$$G_{\mathfrak{a}\mathfrak{b};\Delta} = 2 W_{(\mathfrak{a}\mathfrak{b})\Delta}. \tag{17.10}$$

The equations (17.2) may be regarded as a differential system for the equation $\boldsymbol{X}(\boldsymbol{V})$ of the shell and for the assignment of directors upon it. When $X^{\mathfrak{a}}_{\Delta}$, $G_{\mathfrak{a}\mathfrak{b}} = G_{\mathfrak{b}\mathfrak{a}}$, and $W^{\mathfrak{b}}{}_{\mathfrak{a}\Delta}$ are prescribed as functions of $\boldsymbol{V}$ subject to the conditions (17.8), (17.9), and (17.10), the system (17.2) is completely integrable. Locally there will then exist a unique solution $\boldsymbol{X}(\boldsymbol{V})$ and $\boldsymbol{D}_{\mathfrak{a}}(\boldsymbol{V})$ taking on prescribed values at one value $\boldsymbol{V}^\circ$ of $\boldsymbol{V}$. Provided that not all the quantities $X^{\mathfrak{a}}_{[\Delta} X^{\mathfrak{b}}_{\Xi]}$ vanish, that $G_{\mathfrak{a}\mathfrak{b}}$ be positive definite and that the prescription of $\boldsymbol{D}_{\mathfrak{a}}(\boldsymbol{V}^\circ)$ is consistent with $(17.1)_2$, the solution represents an oriented shell and the relation $(17.1)_1$ holds. By using the uniqueness of solution, it is a simple matter to show that $X^{\mathfrak{b}}_{\Delta}$, $G_{\mathfrak{a}\mathfrak{b}}$, *and* $W^{\mathfrak{b}}{}_{\mathfrak{a}\Delta}$ *determine* $\mathscr{S}$ *and its directors to within a rigid displacement combined with a reflection.*

18. Exact differential description of the strain of a shell. At the points $\boldsymbol{x}$ of the deformed shell s, assign the directors $\boldsymbol{d}_{\mathfrak{a}}$, and, as suggested by (12.1), put

$$x^{\mathfrak{a}}_{\Delta} \equiv d^{\mathfrak{a}}_k x^k_{;\Delta} = d^{\mathfrak{a}}_k x^k_{;\delta} v^\delta_{;\Delta}, \qquad C_{\mathfrak{a}\mathfrak{b}} \equiv g_{km} d^k_{\mathfrak{a}} d^m_{\mathfrak{b}}, \tag{18.1}$$

$$F^{\mathfrak{a}}_{\mathfrak{b}\Delta} \equiv d^{\mathfrak{a}}_k d^k_{\mathfrak{b};\Delta} = d^{\mathfrak{a}}_k d^k_{\mathfrak{b};\delta} v^\delta_{;\Delta},$$

so that

$$x^k_{;\delta} = d^k_{\mathfrak{a}} x^{\mathfrak{a}}_{\Delta} V^{\Delta}_{;\delta}, \qquad d^k_{\mathfrak{b};\delta} = d^k_{\mathfrak{a}} F^{\mathfrak{a}}_{\mathfrak{b}\Delta} V^{\Delta}_{;\delta}. \tag{18.2}$$

The $C_{\mathfrak{a}\mathfrak{b}}$ are the *components of deformation*; $F^{\mathfrak{a}}_{\mathfrak{b}\Delta}$ is the *relative wryness* of s; the $x^{\mathfrak{a}}_{\Delta}$ are certain tangent vectors. Then

$$\begin{aligned} a_{\delta\xi} &= C_{\mathfrak{a}\mathfrak{b}} x^{\mathfrak{a}}_{\Delta} x^{\mathfrak{b}}_{\Xi} V^{\Delta}_{;\delta} V^{\Xi}_{;\xi}, \\ b_{\delta\xi} &= n_k d^k_{\mathfrak{a}} [F^{\mathfrak{a}}_{\mathfrak{b}\Xi} V^{\Xi}_{;\xi} x^{\mathfrak{b}}_{\Delta} V^{\Delta}_{;\delta} + x^{\mathfrak{a}}_{\Delta;\Xi} V^{\Delta}_{;\delta} V^{\Xi}_{;\xi} + x^{\mathfrak{a}}_{\Delta} V^{\Delta}_{;\delta\xi}]. \end{aligned} \tag{18.3}$$

Thus we see that knowledge of $\boldsymbol{v}$, $x^{\mathfrak{a}}_{\Delta}$, $C_{\mathfrak{a}\mathfrak{b}}$, $F^{\mathfrak{a}}_{\mathfrak{b}\Delta}$, and $d^k_{\mathfrak{a}}$ as functions of $\boldsymbol{V}$ suffices to determine the first and second forms of the deformed shell s.

By analogy with (17.8)—(17.10) we have

$$\frac{\partial x^{\mathfrak{a}}_{[\Delta}}{\partial V^{\Xi]}} + F^{\mathfrak{b}}_{\mathfrak{a}[\Xi} X^{\mathfrak{a}}_{\Delta]} = 0, \tag{18.4}$$

$$F^{\mathfrak{c}}_{\mathfrak{b}\Xi} F^{\mathfrak{b}}_{\mathfrak{a}\Delta} - F^{\mathfrak{c}}_{\mathfrak{b}\Delta} F^{\mathfrak{b}}_{\mathfrak{a}\Xi} + 2\frac{\partial F^{\mathfrak{c}}_{\mathfrak{a}[\Delta}}{\partial V^{\Xi]}} = 0, \tag{18.5}$$

$$C_{\mathfrak{a}\mathfrak{b};\Delta} = 2 C_{\mathfrak{c}(\mathfrak{a}} F^{\mathfrak{c}}_{\mathfrak{b})\Delta}. \tag{18.6}$$

Further, when $\boldsymbol{v}$, $F^{\mathfrak{a}}_{\mathfrak{b}\,\Xi}$, $x^{\mathfrak{a}}_{\Delta}$, and $C_{\mathfrak{a}\mathfrak{b}}$ are given, the requirements (18.4)–(18.6), subject to $|v^{\xi}_{;\,\Xi}| \neq 0$, and the conditions that $C_{\mathfrak{a}\mathfrak{b}}$ be symmetric and positive definite and that not all the quantities $x^{\mathfrak{a}}_{[\Delta} x^{\mathfrak{b}}_{\Xi]}$ vanish, the differential system (18.2) determines an oriented shell satisfying (18.1), uniquely to within a rigid motion combined with a reflection. What has been shown, then, is summarized in the following **fundamental theorem on the strain of a shell**: *Given a shell $\mathscr{S}$ with fundamental forms $\boldsymbol{A}(\boldsymbol{V})$ and $\boldsymbol{B}(\boldsymbol{V})$ and with directors $\boldsymbol{D}_{\mathfrak{a}}(\boldsymbol{V})$, prescription of the 32 quantities v^{ξ}, $F^{\mathfrak{a}}_{\mathfrak{b}\,\Xi}$, $x^{\mathfrak{a}}_{\Delta}$ and $C_{\mathfrak{a}\mathfrak{b}}$ as functions of $\boldsymbol{V}$ subject to the aforementioned conditions determines a second shell s with equation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{v})$ and directors $\boldsymbol{d}_{\mathfrak{a}}(\boldsymbol{v})$, uniquely to within a rigid displacement combined with a reflection.* In other words, the quantities v^{ξ}, $F^{\mathfrak{a}}_{\mathfrak{b}\,\Xi}$, $x^{\mathfrak{a}}_{\Delta}$, and $C_{\mathfrak{a}\mathfrak{b}}$ furnish a *complete differential description of the strain of a shell.*

A theory of rotation is easily constructed by analogy to what was done for rods in § 13.

19. Resolution into normal and tangential components. Since $X^{K}_{;1}$, $X^{K}_{;2}$, and N^{K} are three linearly independent vectors, we may write

$$D^{K}_{\mathfrak{a}} = D^{\Delta}_{\mathfrak{a}} X^{K}_{;\,\Delta} + D_{\mathfrak{a}} N^{K}, \tag{19.1}$$

where

$$D^{\Delta}_{\mathfrak{a}} \equiv A^{\Delta\Xi} D^{K}_{\mathfrak{a}} X_{K;\,\Xi}, \qquad D_{\mathfrak{a}} = D^{K}_{\mathfrak{a}} N_{K}. \tag{19.2}$$

Then by (16.5) follows

$$D^{K}_{\mathfrak{a};\,\Delta} = (D^{\Xi}_{\mathfrak{a};\,\Delta} - D_{\mathfrak{a}} B^{\Xi}_{\Delta}) X^{K}_{;\,\Xi} + (D^{\Xi}_{\mathfrak{a}} B_{\Xi\Delta} + D_{\mathfrak{a};\,\Delta}) N^{K}, \tag{19.3}$$

whence

$$N_{K} D^{K}_{\mathfrak{a};\,\Delta} = D^{\Xi}_{\mathfrak{a}} B_{\Xi\Delta} + D_{\mathfrak{a};\,\Delta}, \tag{19.4}$$

$$X_{K;\,\Xi} D^{K}_{\mathfrak{a};\,\Delta} = D_{\mathfrak{a}\,\Xi;\,\Delta} - D_{\mathfrak{a}} B_{\Xi\Delta}. \tag{19.5}$$

From the fact that $B_{\Delta\Xi}$ and $A_{\Delta\Xi}$ determine $\boldsymbol{X}(\boldsymbol{V})$ to within a rigid motion and that $D^{K}_{\mathfrak{a}}(\boldsymbol{V})$ is uniquely determined by $D^{\Delta}_{\mathfrak{a}}$ and $D_{\mathfrak{a}}$ when $\boldsymbol{X}(\boldsymbol{V})$ is known, it follows that $B_{\Delta\Xi}$, $A_{\Delta\Xi}$, $D^{\Delta}_{\mathfrak{a}}$, and $D_{\mathfrak{a}}$ determine $\mathscr{S}$ and its directors to within a rigid motion. Other than (16.4), there are no compatibility conditions to be satisfied by these quantities.

It involves no restriction to require that s and one director not tangent to s, say $\boldsymbol{d}_{I}$, be material with respect to an unspecified three-dimensional deformation. That is, if $\boldsymbol{x}(\boldsymbol{v})$, $\boldsymbol{d}_{I}(\boldsymbol{v})$, $\boldsymbol{v}(\boldsymbol{V})$, $\boldsymbol{X}(\boldsymbol{V})$, and $\boldsymbol{D}_{I}(\boldsymbol{V})$ are given subject to the condition that $\boldsymbol{d}_{I}$ be not tangent to s and $\boldsymbol{D}_{I}$ not tangent to $\mathscr{S}$, there will exist infinitely many mappings $\boldsymbol{x}(\boldsymbol{X})$ such that

$$\boldsymbol{x}(\boldsymbol{v}) = \boldsymbol{x}\big(\boldsymbol{X}(\boldsymbol{V}(\boldsymbol{v}))\big), \quad \boldsymbol{X}(\boldsymbol{V}) = \boldsymbol{X}\big(\boldsymbol{x}(\boldsymbol{v}(\boldsymbol{V}))\big). \tag{19.6}$$

and

$$d^{k}_{I} = x^{k}_{;K} D^{K}_{I}, \tag{19.7}$$

the quantities $x^{k}_{;K}$ being uniquely determined as functions of $\boldsymbol{V}$ by (19.7) and

$$x^{k}_{;\delta} = x^{k}_{;K} X^{K}_{;\Delta} V^{\Delta}_{;\delta}. \tag{19.8}$$

Since the remaining directors can be assigned arbitrarily, they will not necessarily be material with respect to this deformation.

20. *Special case:* SYNGE & CHIEN'S *formalism.* The formalism just given is easily specialized so as to give results depending on particular choices of co-ordinates for the strain of position. We consider here only the most general form of the usual approach, that given by SYNGE & CHIEN [1941, 7]. The intended interpretation is that s and $\boldsymbol{d}_1$ are material, so that (19.6)—(19.8) apply, with $\boldsymbol{D}_1$ varying with the deformation $\boldsymbol{x}(\boldsymbol{X})$ in such a way that $\boldsymbol{d}_1 = \boldsymbol{n}$. Choose co-ordinates and parameters such that

$$\begin{gathered}\boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{X}, \quad \boldsymbol{X}(\boldsymbol{V}) = (V^1, V^2, 0), \quad \boldsymbol{x}(\boldsymbol{v}) = (v^1, v^2, 0), \\ g_{k3} = \delta_{k3}. \end{gathered} \tag{20.1}$$

Then

$$d_1^k = n^k = n_k = \delta_K^k D_1^K = \delta_3^k, \quad N_M = \delta_{M3} (G^{33})^{-\frac{1}{2}}, \tag{20.2}$$

and

$$A_{\Xi\Delta} = G_{KM}\, \delta_\Xi^K\, \delta_\Delta^M, \quad a_{\xi\delta} = g_{km}\, \delta_\xi^k\, \delta_\delta^m. \tag{20.3}$$

From (16.1), $(20.1)_2$, $(20.2)_3$, and $(20.3)_1$

$$\begin{aligned} 2B_{\Xi\Delta} &= 2N_K \left(\frac{\partial^2 X^K}{\partial V^\Xi\, \partial V^\Delta} + \left\{ {K \atop MP} \right\} X^M_{;\Xi} X^P_{;\Delta} + \left\{ {\Lambda \atop \Xi\Delta} \right\} X^K_{;\Lambda} \right), \\ &= 2\, G_{33} \left\{ {3 \atop JK} \right\} \delta_\Xi^J\, \delta_\Delta^K, \\ &= (G^{33})^{-\frac{1}{2}} \left(\frac{\partial G_{33}}{\partial V^\Delta}\, \delta_\Xi^J + \frac{\partial G_{K3}}{\partial V^\Xi}\, \delta_\Delta^K - \frac{\partial G_{JK}}{\partial X^3}\, \delta_\Xi^J\, \delta_\Delta^K \right) + \\ &\quad + (G^{33})^{-\frac{1}{2}}\, G^{3K}\, \delta_K^\Lambda \left(\frac{\partial A_{\Xi\Lambda}}{\partial V^\Delta} + \frac{\partial A_{\Delta\Lambda}}{\partial V^\Xi} - \frac{\partial A_{\Xi\Delta}}{\partial V^\Lambda} \right), \end{aligned} \tag{20.4}$$

which we may write in the form

$$\begin{aligned} \frac{\partial G_{JK}}{\partial X^3}\, \delta_\Xi^J\, \delta_\Delta^K &= \frac{\partial G_{J3}}{\partial V^\Delta}\, \delta_\Xi^J + \frac{\partial G_{J3}}{\partial V^\Xi}\, \delta_\Delta^J + \\ &\quad + (G^{33})^{-1}\, G^{3K}\, \delta_K^\Lambda \left(\frac{\partial A_{\Xi\Lambda}}{\partial V^\Delta} + \frac{\partial A_{\Delta\Lambda}}{\partial V^\Xi} - \frac{\partial A_{\Xi\Delta}}{\partial V^\Lambda} \right) - 2\, (G^{33})^{-\frac{1}{2}}\, B_{\Xi\Delta}. \end{aligned} \tag{20.5}$$

This is essentially SYNGE & CHIEN'S equation (69). *Cf.* also CHIEN [1944, eq. (6.13)].

From (19.2) and (20.2),

$$D_{1\Delta} = G_{K3}\, \delta_\Delta^K, \quad D_{13} = (G_{33})^{-\frac{1}{2}}. \tag{20.6}$$

Now, using $(20.1)_1$, $(20.2)_1$, and (20.6), we obtain

$$\begin{aligned} 2X_{K;\Xi}\, D^K_{1;\Delta} &= 2G_{JK}\, X^J_{;\Xi} \left(\frac{\partial D_1^K}{\partial V^\Delta} + \left\{ {K \atop MP} \right\} X^M_{;\Delta}\, D_1^P \right), \\ &= 2G_{JK}\, \delta_\Xi^J \left\{ {K \atop MP} \right\} \delta_\Delta^M\, \delta_3^P, \\ &= \frac{\partial G_{JK}}{\partial X^3}\, \delta_\Xi^J\, \delta_\Delta^K + \frac{\partial G_{J3}}{\partial V^\Delta}\, \delta_\Xi^J - \frac{\partial G_{K3}}{\partial V^\Xi}\, \delta_\Delta^K. \end{aligned} \tag{20.7}$$

From (19.5), (20.6), and (20.7) we have

$$
\begin{aligned}
2X_{K;(\Xi}D^K_{I;\Delta)} &= \frac{\partial G_{JK}}{\partial X^3}\,\delta^J_\Xi\,\delta^K_\Delta = D_{I\Xi;\Delta} + D_{I\Delta;\Xi} - 2D_I\,B_{\Xi\Delta}, \\
&= \frac{\partial D_{I\Xi}}{\partial V^\Delta} + \frac{\partial D_{I\Delta}}{\partial V^\Xi} - 2D_{I\Lambda}\left\{{}^{\Lambda}_{\Xi\Delta}\right\} - 2D_I\,B_{\Xi\Delta}, \qquad (20.8)\\
&= \frac{\partial G_{J3}}{\partial V^\Delta}\,\delta^J_\Xi + \frac{\partial G_{J3}}{\partial V^\Xi}\,\delta^J_\Delta - G_{J3}\,\delta^J_\Lambda\,A^{\Lambda\Sigma}\left(\frac{\partial A_{\Xi\Sigma}}{\partial V^\Delta} + \frac{\partial A_{\Delta\Sigma}}{\partial V^\Xi} - \frac{\partial A_{\Xi\Delta}}{\partial V^\Sigma}\right) - 2(G^{33})^{-\frac{1}{2}}\,B_{\Xi\Delta},
\end{aligned}
$$

which is similar to, but not identical with (20.5). The apparent discrepancy is resolved by noting that the equations

$$A^{\Xi\Lambda}A_{\Lambda\Delta} = \delta^\Xi_\Delta, \qquad 0 = \delta^3_\Delta = G^{3K}\,G_{KJ}\,\delta^J_\Delta = G^{33}\,G_{3J}\,\delta^J_\Delta + G^{3K}\,\delta^\Xi_K\,A_{\Xi\Delta}, \qquad (20.9)$$

imply that

$$(G^{33})^{-1}\,G^{3K}\,\delta^\Xi_K = -\,G_{3J}\,\delta^J_\Delta\,A^{\Delta\Xi}. \qquad (20.10)$$

For $\mathscr{s}$, we have by analogy with (20.5)

$$2b_{\alpha\beta} = \frac{\partial g_{km}}{\partial x^3}\,\delta^k_\alpha\,\delta^m_\beta = \frac{\partial g_{km}}{\partial X^3}\,\delta^k_\alpha\,\delta^m_\beta. \qquad (20.11)$$

SYNGE & CHIEN found it sufficient to introduce nine measures of strain. The set $a_{\alpha\beta}$, $b_{\alpha\beta}$, $D_{I\Delta}$, and D_I is equivalent to that which they used. These satisfy three compatibility conditions which may be taken as the dual* of (16.4). When $\mathscr{S}$ is given, these quantities determine $\boldsymbol{D}_I$ uniquely and $\mathscr{s}$ to within a rigid motion. With the conventions adopted above, part of the problem involves determining a tensor g_{km} consistent with the conditions $(20.3)_1$, (20.11), and the fact that the Riemann tensor based on g_{km} must vanish; such complexity is the price one pays for eliminating the functions $\boldsymbol{x}(\boldsymbol{v})$ as unknowns.

Part III. Theory of stress

A. Rods

21. Stress principle for rods. A rod will be regarded here simply as a curve c which may be the seat of dynamical actions. The basic postulate is the **stress principle:** *At each point on a rod, the action of the material to one side upon the material to the other is equipollent to that of a* stress resultant vector $\boldsymbol{S}$ *and a* couple resultant $\boldsymbol{M}$. If we denote by $\boldsymbol{S}_+$ and $\boldsymbol{M}_+$ the values of the vectors $\boldsymbol{S}$ and $\boldsymbol{M}$ appropriate to the action of the material to one side of a point s on that to the other, and by $\boldsymbol{S}_-$ and $\boldsymbol{M}_-$ the values of $\boldsymbol{S}$ and $\boldsymbol{M}$ when the roles of the two sides are interchanged, then the balance of forces and moments requires that

$$\boldsymbol{S}_+ = -\,\boldsymbol{S}_-, \qquad \boldsymbol{M}_+ = -\,\boldsymbol{M}_-. \qquad (21.1)$$

Thus, with an appropriate convention of sign, we may use the vectors $\boldsymbol{S}$ and $\boldsymbol{M}$ without further designation.

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* These equations are equivalent to the three given by SYNGE & CHIEN [1941, Eqs. (38), (39)].

22. Differential equations of equilibrium in space. Let $\boldsymbol{F}$ and $\boldsymbol{L}$, assumed to be integrable functions of s, denote the assigned force and couple per unit length acting upon the rod. Then for points where $\boldsymbol{S}$ and $\boldsymbol{M}$ are differentiable and $\boldsymbol{F}$ and $\boldsymbol{L}$ are bounded, a local statement of the principle of equilibrium is

$$\begin{gathered} \check{\boldsymbol{S}} + \boldsymbol{F} = 0, \\ \check{\boldsymbol{M}} + \overline{\boldsymbol{p} \times \boldsymbol{S}}^{\vee} + \boldsymbol{L} + \boldsymbol{p} \times \boldsymbol{F} = 0, \end{gathered} \tag{22.1}$$

where $\boldsymbol{p}$ is the position vector to s from the origin of some inertial frame and where "$\smile$" denotes the intrinsic derivative, defined as in (8.3), except that s, the arc length *in the actual configuration of the rod*, is the independent variable*. By substituting $(22.1)_1$ into $(22.1)_2$ we may eliminate $\boldsymbol{F}$ and replace $(22.1)_2$ by the equivalent condition

$$\check{\boldsymbol{M}} + \boldsymbol{t} \times \boldsymbol{S} + \boldsymbol{L} = 0, \tag{22.2}$$

where $\boldsymbol{t}$ is the unit tangent. At points where $\boldsymbol{S}$ and $\boldsymbol{M}$ are not differentiable, (22.1) and (22.2) need not hold, but application of the principle of equilibrium shows that $\boldsymbol{S}$ and $\boldsymbol{M}$ must be continuous if $\boldsymbol{F}$ and $\boldsymbol{L}$ are bounded**.

23. Resolution into normal and tangential components. Since (22.1) and (22.2) are in vectorial form, they are valid in an arbitrary curvilinear co-ordinate system. It is customary, however, to refer them to a particular frame defined with respect to the rod c. Retaining full generality at the start, in the scheme of § 12 we assign three linearly independent directors $\boldsymbol{d}_\mathfrak{a}$ and reciprocal directors $\boldsymbol{d}^\mathfrak{a}$ to c. In general three-dimensional co-ordinates, let S^k and F^k be the contravariant components of $\boldsymbol{S}$ and $\boldsymbol{F}$, M_k and L_k the covariant components of $\boldsymbol{M}$ and $\boldsymbol{L}$, and define corresponding anholonomic components:

$$\begin{aligned} S^\mathfrak{a} &= d^\mathfrak{a}_k S^k, & M_\mathfrak{a} &= d^k_\mathfrak{a} M_k, \\ F^\mathfrak{a} &\equiv d^\mathfrak{a}_k F^k, & L_\mathfrak{a} &\equiv d^k_\mathfrak{a} L_k. \end{aligned} \tag{23.1}$$

From $(22.1)_1$ and the result dual*** to the reciprocal of $(9.1)_3$ we have

$$\begin{aligned} \frac{dS^\mathfrak{a}}{ds} = \check{S}^\mathfrak{a} &= d^\mathfrak{a}_k \check{S}^k + \check{d}^\mathfrak{a}_k S^k, \\ &= - d^\mathfrak{a}_k F^k - d^\mathfrak{a}_m w^m{}_k \, S^k, \end{aligned} \tag{23.2}$$

where $\boldsymbol{w}$ is the wryness of the directors along c. Similarly

$$\frac{dM_\mathfrak{a}}{ds} = \check{M}_\mathfrak{a} = - d^k_\mathfrak{a} (e_{kpq} t^p S^q + L_k) + d^m_\mathfrak{a} w^k{}_m M_k. \tag{23.3}$$

* "$\smile$" is the dual of "$\frown$".

** If concentrated loads are present, they determine jumps in $\boldsymbol{S}$ and $\boldsymbol{M}$:

$$[\boldsymbol{S}] = - \boldsymbol{F}_c, \quad [\boldsymbol{M}] = - \boldsymbol{L}_c.$$

*** Note that $\boldsymbol{w}$ is not the relative wryness $\boldsymbol{F}$ of § 12 but rather the dual of the wryness $\boldsymbol{W}$ of § 9.

Hence the statical equations in anholonomic components are*

$$\frac{dS^{\mathfrak{a}}}{ds} + w^{\mathfrak{a}}_{\ \mathfrak{b}}\, S^{\mathfrak{b}} + F^{\mathfrak{a}} = 0,$$
$$\frac{dM_{\mathfrak{a}}}{ds} - w^{\mathfrak{b}}_{\ \mathfrak{a}}\, M_{\mathfrak{b}} + e_{\mathfrak{a}\mathfrak{b}\mathfrak{c}}\, t^{\mathfrak{b}}\, S^{\mathfrak{c}} + L_{\mathfrak{a}} = 0. \tag{23.4}$$

Thus far the director frame has been arbiirary. We now require it to be a unit orthogonal triad such that $\boldsymbol{d}_1 = \boldsymbol{t}$, the unit tangent. By the dual of $(9.2)_3$, the wryness $\boldsymbol{w}$ then satisfies $w_{\mathfrak{a}\mathfrak{b}} = -w_{\mathfrak{b}\mathfrak{a}}$ and may be interpreted as an angular velocity**. The component S^1, which is the projection of $\boldsymbol{S}$ onto the tangent to c, is called the specific *tension* of the rod; the components S^2 and S^3, the specific *shearing forces*; M^1, the specific *twisting couple*; M^2 and M^3, the specific *bending couples*. This special choice of directors, while not simplifying $(23.4)_1$, implies that $t^1 = 1$, $t^2 = t^3 = 0$ and hence reduces the three components of $(23.4)_2$ to the following explicit forms:

$$\frac{dM_1}{ds} - w^{\mathfrak{b}}_{\ 1}\, M_{\mathfrak{b}} + L_1 = 0,$$
$$\frac{dM_2}{ds} - w^{\mathfrak{b}}_{\ 2}\, M_{\mathfrak{b}} - S^3 + L_2 = 0, \tag{23.5}$$
$$\frac{dM_3}{ds} - w^{\mathfrak{b}}_{\ 3}\, M_{\mathfrak{b}} + S^2 + L_3 = 0.$$

B. Shells

24. Stress principle for shells. A shell will be regarded here simply as a surface s which may be the seat of dynamical actions. The **stress principle for shells** asserts that *the action of the part of the shell outside any imagined closed curve c on the part inside is equipollent to a field of* stress resultant vectors $\boldsymbol{S}_{(\boldsymbol{n})}$ *and* couple resultant vectors $\boldsymbol{M}_{(\boldsymbol{n})}$ *defined on c.* The subscript*** $\boldsymbol{n}$ refers to the

* By definition

$$e_{ijk} = +\sqrt{g}\, \varepsilon_{ijk},$$

where ε_{ijk} is the permutation symbol such that $\varepsilon_{123} = +1$. Hence

$$e_{\mathfrak{a}\mathfrak{b}\mathfrak{c}} = +\sqrt{g}\, \varepsilon_{kpq}\, d^k_{\mathfrak{a}}\, d^p_{\mathfrak{b}}\, d^q_{\mathfrak{c}},$$
$$= +\sqrt{g}\, \varepsilon_{\mathfrak{a}\mathfrak{b}\mathfrak{c}} \det d^k_{\mathfrak{e}}.$$

Now

$$g(\det d^k_{\mathfrak{e}})^2 = \det(g_{km}\, d^k_{\mathfrak{e}}\, d^m_{\mathfrak{f}}) = \det g_{\mathfrak{e}\mathfrak{f}},$$

hence

$$e_{\mathfrak{a}\mathfrak{b}\mathfrak{c}} = \pm\sqrt{\det g_{\mathfrak{e}\mathfrak{f}}}\, \varepsilon_{\mathfrak{a}\mathfrak{b}\mathfrak{c}},$$

where the sign is so selected so as to agree with that of $\det d^k_{\mathfrak{a}}$. The quantity $\det g_{\mathfrak{e}\mathfrak{f}}$ has been evaluated in the footnote on p. 301. In particular, for directors forming a right-handed orthogonal unit triad we thus obtain

$$e_{\mathfrak{a}\mathfrak{b}\mathfrak{c}} = \varepsilon_{\mathfrak{a}\mathfrak{b}\mathfrak{c}}.$$

** The classical notation for the component w_{23} is τ or $-\tau$; the other two independent components are written as $\pm\varkappa$ and $\pm\varkappa'$. In the classical treatments it is not always made clear that these quantities refer to the *loaded* rod.

*** This $\boldsymbol{n}$ is not to be confused with the unit normal to s, which was denoted by $\boldsymbol{n}$ in §§ 16—20, as it will again from § 26 onward.

unit outward normal to c; of course, $\boldsymbol{n}$ is a vector field defined intrinsically in the surface s, but $\boldsymbol{S}_{(\boldsymbol{n})}$ and $\boldsymbol{M}_{(\boldsymbol{n})}$ are fields defined in the three-dimensional Euclidean space in which s is embedded. A mathematical expression of the postulated principle, for the case of equilibrium, is

$$\begin{gathered}\oint_c \boldsymbol{S}_{(\boldsymbol{n})}\, ds + \int_{s_1} \boldsymbol{F}\, da = 0,\\ \oint_c (\boldsymbol{M}_{(\boldsymbol{n})} + \boldsymbol{p}\times\boldsymbol{S}_{(\boldsymbol{n})})\, ds + \int_{s_1} (\boldsymbol{L} + \boldsymbol{p}\times\boldsymbol{F})\, da = 0,\end{gathered} \tag{24.1}$$

where s is arc length along c, where s_1 is the portion of the surface s inclosed by c, where $\boldsymbol{p}$ is the three-dimensional position vector to the running point of integration, and where $\boldsymbol{F}$ and $\boldsymbol{L}$ are the three-dimensional fields of assigned force and assigned couple. The vector integrals in (24.1) are to be understood as abbreviations for the integrals of rectangular Cartesian components.

A classical argument, not different in principle from its counterpart for three-dimensional bodies, yields not only

$$\boldsymbol{S}_{(\boldsymbol{n})} = -\boldsymbol{S}_{(-\boldsymbol{n})}, \qquad \boldsymbol{M}_{(\boldsymbol{n})} = -\boldsymbol{M}_{(-\boldsymbol{n})} \tag{24.2}$$

but also

$$S^k_{(\boldsymbol{n})} = S^{k\delta} n_\delta, \qquad M^k_{(\boldsymbol{n})} = M^{k\delta} n_\delta, \tag{24.3}$$

where, as in all that follows, Greek indices have the range 1, 2. The quantities n_δ are the covariant components of the unit normal to c in *any curvilinear co-ordinate system* v^1, v^2 on s. By hypothesis, the quantities $\boldsymbol{S}_{(\boldsymbol{n})}$ are absolute vectors and the $\boldsymbol{M}_{(\boldsymbol{n})}$ are axial vectors; while to derive (24.3) rectangular Cartesian spatial co-ordinates were adopted, the results are double tensor equations in the sense defined in § 3 and hence are valid in all spatial co-ordinates. The double tensors $S^{k\delta}$ and $M^{k\delta}$ are the fields of *stress resultants* and *stress couples*. There are six components $S^{k\delta}$ and six components $M^{k\delta}$.

In the classical treatments of the theory of shells, the vectors $\boldsymbol{L}$ and $\boldsymbol{M}$ are assumed tangent to s; in this case the number of independent components $M^{k\delta}$ is reduced from six to four. (Roughly speaking, this assumption corresponds to the case when the assigned couples result only from forces acting a finite distance from s, not from internal couples such as occur in polarized media.) In mathematical form this assumption asserts the existence of surface tensors L^δ and $M^{\xi\delta}$ such that

$$L^k = x^k_{;\delta} L^\delta, \qquad M^{k\xi} = x^k_{;\delta} M^{\delta\xi}, \tag{24.4}$$

where $x^k_{;\delta} = \partial x^k/\partial v^\delta$, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{v})$ being an equation of s in general co-ordinates.

25. *Differential equations of equilibrium in space.* Again we suppose the space co-ordinates rectangular Cartesian, we consider $\boldsymbol{S}$ and $\boldsymbol{M}$ as functions of $\boldsymbol{v}$ only, and we substitute (24.3) into (24.1). In regions where $\boldsymbol{S}$ and $\boldsymbol{M}$ are continuously differentiable, a classical argument based on GREEN's transformation yields the differential equations

$$\begin{gathered} S^{k\delta}{}_{,\delta} + F^k = 0,\\ \overline{M}^{kp\delta}{}_{,\delta} + z^{[k}{}_{,\delta} S^{p]\delta} + \overline{L}^{kp} = 0,\end{gathered} \tag{25.1}$$

where $\overline{M}^{kp\delta}$ and $\overline{L}^{kp}$ are absolute alternating tensors equivalent to the axial vectors $M^{k\delta}$ and L^k, and where the subscript comma denotes the covariant derivative with respect to the surface metric $\boldsymbol{a}$, except that $z^k{}_{,\delta} \equiv \partial z^k/\partial v^\delta$, $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{v})$ being a rectangular Cartesian equation of the surface s.

Now consider the equations

$$\begin{aligned} S^{k\delta}{}_{;\delta} + F^k &= 0, \\ \overline{M}^{kp\delta}{}_{;\delta} + x^{[k}{}_{;\delta}\, S^{p]\delta} + \overline{L}^{kp} &= 0, \end{aligned} \tag{25.2}$$

where the space co-ordinates and the surface co-ordinates are *arbitrary independently selected general curvilinear systems,* where $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{v})$ is an equation of s referred to these two systems, where $x^k{}_{;\delta} = \partial x^k / \partial v^\delta$, and where other occurrences of the semicolon denote the total covariant derivative (3.3). The equations (25.2) are in double tensor form; when the space co-ordinates are rectangular Cartesian, (25.2) reduce to (25.1), already proved valid in such systems; therefore (25.2) are the **general differential equations of equilibrium for shells**. We may continue to regard $\boldsymbol{S}$ and $\boldsymbol{M}$ as functions of $\boldsymbol{v}$ only, or we may consider them to be functions of $\boldsymbol{x}$ also, as we please.

The elegant simplicity of this derivation should not conceal the complexity of the result. When $(25.2)_1$ is written out, it assumes the form

$$\frac{\partial S^{k\delta}}{\partial v^\delta} + \left\{ {k \atop m\,p} \right\} x^p_{;\delta}\, S^{m\delta} + \left\{ {\xi \atop \delta\,\xi} \right\} S^{k\delta} + F^k = 0, \tag{25.3}$$

where $\left\{ {k \atop m\,p} \right\}$ and $\left\{ {\xi \atop \delta\,\xi} \right\}$ are Christoffel symbols based upon the space metric $\boldsymbol{g}$ and the surface metric $\boldsymbol{a}$, respectively, the two metrics being related as usual: $a_{\delta\xi} = g_{km}\, x^k_{;\delta}\, x^m_{;\xi}$. When $(25.2)_2$ is written explicitly in terms of the axial vector $M^{k\delta}$, it assumes the form

$$\frac{\partial M^{k\delta}}{\partial v^\delta} + \left\{ {k \atop m\,p} \right\} x^p_{;\delta}\, M^{m\delta} + \left\{ {\xi \atop \delta\,\xi} \right\} M^{k\delta} + e^k_{\ m\,p}\, x^m_{;\delta}\, S^{p\delta} + L^k = 0, \tag{25.4}$$

where $e^k_{\ m\,p} = g^{ks} \sqrt{g}\, \varepsilon_{smp}$, ε_{smp} being the permutation symbol such that $\varepsilon_{123} = +1$. These formulae involve doubly contravariant tensors considered as functions of $\boldsymbol{v}$ only. In terms of physical components or of components allowed to depend on $\boldsymbol{x}$ as well, they would take on still more complicated forms. The results (25.3) and (25.4), specialized by means of the assumption (24.4), are the differential equations of equilibrium in the general form first derived by SYNGE & CHIEN [1941, pp. 104–111].

26. Resolution into normal and tangential components. Resolving the variables occurring in (25.2), we may write

$$\begin{aligned} F^k &= F^\delta\, x^k_{;\delta} + F\, n^k, & L^k &= L^\delta\, x^k_{;\delta} + L\, n^k, \\ S^{k\delta} &= S^{\gamma\delta}\, x^k_{;\gamma} + S^\delta\, n^k, & M^{k\delta} &= M^{\gamma\delta}\, x^k_{;\gamma} + M^\delta\, n^k, \end{aligned} \tag{26.1}$$

where $\boldsymbol{n}$ is the unit normal to s. The following table connects the components occurring in (26.1) with the terms usually employed in shell theory:

F, L = *normal components of specific applied force and couple.*
F^δ, L^δ = *specific applied force and couple tangent to the shell.*
S^δ = *cross force resultant.*
$S^{\gamma\delta}$ = *membrane stress resultant.*
M^δ = *cross moment resultant.*
$M^{\delta\gamma}$ = *couple resultant.*

In this notation, the classical assumption (24.4) takes the form

$$L = 0, \qquad M^{\delta} = 0, \tag{26.2}$$

but this we do not at once adopt. The normal and shear components of $S^{\gamma\delta}$ in an orthogonal co-ordinate system are called *normal* and *shear* membrane stress resultants; the normal and shear components of $M^{\gamma\delta}$ in such a system are called *twisting* and *bending* couple resultants, respectively.

To express the equations of equilibrium in terms of tangential and normal components*, we use the identities

$$\begin{aligned} e_{\gamma\delta}\, n_q &= e_{q k p}\, x^k_{;\delta}\, x^p_{;\gamma}, \qquad & e^q{}_{kp}\, x^k_{;\delta}\, n^p &= a^{\gamma\lambda}\, e_{\lambda\delta}\, x^q_{;\gamma}, \\ x^k_{;\gamma\delta} &= b_{\gamma\delta}\, n^k, & n^k_{;\gamma} &= -\, b_{\gamma\delta}\, a^{\delta\sigma}\, x^k_{;\sigma}, \end{aligned} \tag{26.3}$$

where $\boldsymbol{b}$ is the second fundamental form of s. From (26.1) and these identities follows

$$S^{k\delta}{}_{;\delta} = (S^{\gamma\delta}{}_{;\delta} - a^{\sigma\gamma}\, b_{\sigma\delta}\, S^{\delta})\, x^k_{;\gamma} + (S^{\delta}{}_{;\delta} + b_{\gamma\delta}\, S^{\gamma\delta})\, n^k. \tag{26.4}$$

Substituting this result and $(26.1)_1$ into $(25.2)_1$ yields

$$(S^{\gamma\delta}{}_{;\delta} - a^{\sigma\gamma}\, b_{\sigma\delta}\, S^{\delta} + F^{\delta})\, x^k_{;\gamma} + (S^{\delta}{}_{;\delta} + b_{\gamma\delta}\, S^{\gamma\delta} + F)\, n^k = 0. \tag{26.5}$$

Taking the scalar product of this equation by $\boldsymbol{n}$ yields the *condition for equilibrium of normal forces:*

$$S^{\delta}{}_{;\delta} + b_{\gamma\delta}\, S^{\gamma\delta} + F = 0; \tag{26.6}$$

taking the vector product by $\boldsymbol{n}$, the *conditions for equilibrium of tangential forces:*

$$S^{\gamma\delta}{}_{;\delta} - a^{\gamma\sigma}\, b_{\sigma\delta}\, S^{\delta} + F^{\delta} = 0. \tag{26.7}$$

Similar resolution of $(25.2)_2$ yields the *condition for equilibrium of twisting moments:*

$$M^{\delta}{}_{;\delta} + b_{\gamma\delta}\, M^{\gamma\delta} + e_{\gamma\delta}\, S^{\gamma\delta} + L = 0 \tag{26.8}$$

and the *conditions for equilibrium of bending moments:*

$$M^{\gamma\delta}{}_{;\delta} - a^{\gamma\sigma}\, b_{\sigma\delta}\, M^{\delta} + a^{\gamma\delta}\, e_{\delta\sigma}\, S^{\sigma} + L^{\gamma} = 0. \tag{26.9}$$

In these formulae it is legitimate and natural to regard all fields as functions of the surface co-ordinates $\boldsymbol{v}$ only; in this interpretation, the double covariant derivatives ";" reduce to ",", the usual covariant differentiation based in the surface metric $\boldsymbol{a}$. The fully general equations (26.6)—(26.9), but in rectangular Cartesian spatial co-ordinates, were first given by E. & F. Cosserat [1909, 1, §§ 35—37], who derived forms in material co-ordinates as well. *Cf.* also Heun [1913, § 20].

Under the classical assumptions (26.2) the total number of independent components of $\boldsymbol{S}$ and $\boldsymbol{M}$ is reduced from 12 to 10, and the equations (26.8) and

* This resolution is effected by Synge & Chien [1941, p. 109] by use of a special co-ordinate system.

(26.9) for equilibrium of moments reduce to

$$\begin{gathered} b_{\gamma\delta} M^{\gamma\delta} + e_{\gamma\delta} S^{\gamma\delta} = 0, \\ M^{\gamma\delta}{}_{;\delta} + a^{\gamma\delta} e_{\delta\sigma} S^{\sigma} + L^{\gamma} = 0. \end{gathered} \tag{26.10}$$

The first of these is a linear algebraic equation expressing the difference of shear resultants, $S^{[12]}$, as a linear combination of the four couple resultants $M^{\gamma\delta}$.

Further specializations appropriate to special co-ordinate systems on s or to special dynamical assumptions occupy much of the literature and are not discussed here.

27. Remarks on the relation of stress and couple resultants to stresses, and on the equations of motion. It is also legitimate to treat shells and rods as three-dimensional bodies and to regard the stress resultants and couple resultants as derivative quantities, *defined* in terms of the three-dimensional stress tensor as certain fields existing only on the line or surface but equipollent there to the field of internal stress in the whole shell or rod. For the theory of plates, this view goes back to CAUCHY and POISSON, but for curved shells it was first carried through exactly and in some generality by NOVOZHILOV [1943, 1, § 1] (*cf.* also NOVOZHILOV & FINKELSTEIN [1943, 2, §§ 1, 4]), by TRUESDELL [1945, § 8], and by CHIEN [1948]*. We do not present a derivation by this method because such a derivation in general co-ordinates has been given ZERNA [1949, § 3] (*cf.* also GREEN & ZERNA [1950, § 3] [1954, 2, § 10.2]). The method consists in integrating the three-dimensional equations across the shell; of course, no approximation is involved. The result are formally identical with (26.6), (26.7), and (26.10).

The interest in a derivation of this kind, beyond its great unifying value, is twofold. First, the applied force and couple $\boldsymbol{F}$ and $\boldsymbol{L}$, which are necessarily quantities taken *a priori* in the direct theory given in this paper, appear as quantities defined in terms not only of the applied internal force but also the applied surface load, and in an exact theory the thickness and curvature of the shell have their influence on the net effect of these loads as it appears in the fields $\boldsymbol{F}$ and $\boldsymbol{L}$ defined on s.

Second, this method enables us to obtain equations of motion, since all that is required is to replace the *three-dimensional* applied force $\boldsymbol{f}$ per unit mass by $\boldsymbol{f} - \boldsymbol{a}$, where $\boldsymbol{a}$ is the *three-dimensional* acceleration. No such simple device can be applied directly to the shell or rod; while, indeed, an element of the shell or rod is endowed with momentum, in an exact theory this is generally *not* proportional to the velocity of any point on the shell or rod. This distinction, if properly pondered, will convince the reader (if he has any doubts) that in truth, occasional conversation to the contrary, at bottom we always conceive the world as having three dimensions at least. Without such a conception, the distinction just made would not have meaning, but without this distinction we should be unable, in principle, to determine the exact motion of points on a shell from the exact dynamical equations.

* All this work appears to date from the period 1942—1944 and to be done independently. The proper *definitions* of stress resultants and couple resultants in terms of the three-dimensional stress tensor are due to LOVE [1893, 2, § 339]; the essential idea was given by LAMB [1890, 1, § 2] and BASSET [1890, 2, §§ 5, 18].

For rods, an exact derivation based on the three-dimensional equations of equilibrium has never been worked out*. It should be equally possible to base such a derivation on the exact equations for shells.

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* While older authors (*e.g.*, Love [1906, § 254]) have given definitions of the stress resultants and couple resultants which are doubtless correct, their subsequent arguments, as in their treatments of shells, rest on unnecessary and unrigorous approximations and limits instead of exact integration.

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On the Riemann-Green Function

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§ 1. Introduction

The first general solution of the problem of Cauchy for an extensive class of partial differential equations was given by Riemann almost a century ago in his well-known paper on the propagation of sound waves of finite amplitude*. Although stated only for certain special equations, it is applicable to any linear equation of hyperbolic type of the second order in two independent variables; it depends ultimately on finding a certain subsidiary function, often called the Riemann-Green function, which is the solution of a characteristic boundary value problem for the adjoint equation. Riemann gave explicit formulae for this subsidiary function in two cases of special importance in gas dynamics.

Although a full account of the method in its general form was given by Darboux**, little progress has been made, and this seems to be due to the difficulty of finding the Riemann-Green function. The books often content themselves either with stating and verifying an expression for the function in one or two cases or with finding it in special cases by an inspired guess as to its form. It seems then to be worth while reviewing the present position in the hope of making Riemann's method more generally useful.

§ 2. A description of Riemann's method

A linear hyperbolic equation of the second order in two independent variables may be put into the form

$$L\,U \equiv \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + 2a\frac{\partial U}{\partial x} - 2b\frac{\partial U}{\partial y} + c\,U = 0 \tag{2.1}$$

by choosing the coordinates x and y so that the two families of characteristics are $x \pm y =$ constant; the coefficients a, b, c are functions of x and y alone. The problem of Cauchy is to find a solution U, given the values of U and its first derivatives on a certain curve C which has the property that no characteristic cuts it in more than one point. Under certain continuity conditions, the problem has a unique solution, given by

$$\begin{aligned} U(X,Y) = {} & \tfrac{1}{2}[U\,V]_A + \tfrac{1}{2}[U\,V]_B + \\ & + \tfrac{1}{2}\int_{AB}\{(V U_y - U V_y + 2b\,U\,V)\,dx + (V U_x - U V_x + 2a\,U\,V)\,dy\} \end{aligned} \tag{2.2}$$

* Riemann: Abh. d. Kön. Ges. der Wiss. zu Göttingen **8** (1860), reprinted in Collected Works of Bernard Riemann, pp. 156—175. Dover Press 1953.

** Darboux: Leçons sur la Théorie des Surfaces, vol. II, pp. 71—111. Paris 1915.

where A and B are the points in which C is cut by the characteristics through $P(X, Y)$; $V \equiv V(x, y; X, Y)$, the Riemann-Green function, is the solution of the adjoint equation

$$L^* V \equiv \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} - 2\frac{\partial}{\partial x}(aV) + 2\frac{\partial}{\partial y}(bV) + cV = 0 \tag{2.3}$$

such that

$$\begin{aligned} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} &= (a+b)\, V \qquad \text{on } y - x = Y - X, \\ \frac{\partial V}{\partial x} - \frac{\partial V}{\partial y} &= (a-b)\, V \qquad \text{on } y + x = Y + X, \\ V &= 1 \qquad \text{at } (X, Y). \end{aligned} \tag{2.4}$$

It is readily seen that

$$\iint \left\{ \frac{\partial}{\partial x}(VU_x - UV_x + 2aUV) - \frac{\partial}{\partial y}(VU_y - UV_y + 2bUV) \right\} dx\, dy$$

vanishes for any region of integration. If we take the region to be that bounded by the characteristics PA, PB and the arc AB of the curve C, equation (2.2) follows at once by GREEN's transformation. (See Fig. 1.)

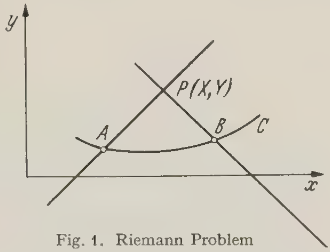

Fig. 1. Riemann Problem

The proof of the existence of this Riemann-Green function $V(x, y; X, Y)$ is omitted as we shall be concerned with its actual construction in certain special cases. For the moment, we note the following properties.

(*i*) V does not depend on the choice of the curve C.

(*ii*) If we change the dependent variable in (2.1) from U to U_1 where $U_1 = \varphi(x, y)\, U$, the Riemann-Green function for the transformed equation is

$$V_1(x, y; X, Y) = \frac{\varphi(X, Y)}{\varphi(x, y)} V(x, y; X, Y). \tag{2.5}$$

(*iii*) The function $U(x, y; X, Y) \equiv V(X, Y; x, y)$ is the Riemann-Green function of the adjoint equation, that is *qua* function of (x, y), it satisfies the equation $LU = 0$ and the conditions

$$\begin{aligned} \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} &= -(a+b)\, U \qquad \text{on } y - x = Y - X, \\ \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} &= -(a-b)\, U \qquad \text{on } y + x = Y + X, \\ U &= 1 \qquad \text{at } (X, Y). \end{aligned}$$

Sometimes it is more convenient to use the characteristic variables

$$r = x - y, \qquad s = x + y.$$

The standard form of the equation is then

$$Lu = \frac{\partial^2 u}{\partial r\, \partial s} + a' \frac{\partial u}{\partial r} + b' \frac{\partial u}{\partial s} + c' u = 0 \tag{2.6}$$

where a', b', c' are functions of r and s. In terms of these variables, RIEMANN'S formula for the solution of the problem of Cauchy is

$$u(R,S) = \tfrac{1}{2}[u\,v]_A + \tfrac{1}{2}[u\,v]_B - \\ -\tfrac{1}{2}\int_{AB}\{(v\,u_r - u\,v_r + 2b'u\,v)\,dr - (v\,u_s - u\,v_s + 2a'u\,v)\,ds\} \tag{2.7}$$

where A and B are the points in which C is cut by the characteristics $r=R$ and $s=S$ respectively through $P(R,S)$; $v \equiv v(r,s;R,S)$ is the Riemann-Green function, the solution of the adjoint equation

$$L^*v \equiv \frac{\partial^2 v}{\partial r\,\partial s} - \frac{\partial}{\partial r}(a'v) - \frac{\partial}{\partial s}(b'v) + c'v = 0 \tag{2.8}$$

such that

$$\begin{aligned} \frac{\partial v}{\partial s} &= a'v && \text{on } r=R, \\ \frac{\partial v}{\partial r} &= b'v && \text{on } s=S, \\ v &= 1 && \text{at } (R,S) \end{aligned} \tag{2.9}$$

(See Fig. 2.)

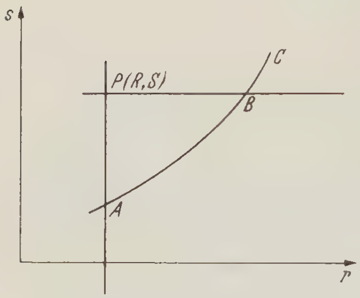

Fig. 2. Riemann Problem

The two definitions are equivalent, in that V is transformed into v by the change to characteristic variables, and conversely.

So far as I know, six ways have been used to find the Riemann-Green function for particular types of hyperbolic equations. These will be discussed in the following order.

(*i*) RIEMANN'S original method was based on the fact that the Riemann-Green function does not depend in any way on the curve carrying the Cauchy data. If it is possible to solve by some other means the Problem of Cauchy for a special curve C depending on one variable parameter, a comparison of the two solutions should give the Riemann-Green function. In the case of the two equations considered by RIEMANN, it was possible to solve the Problem of Cauchy by a FOURIER cosine transform with Cauchy data on a straight line.

(*ii*) HADAMARD pointed out* that the coefficient of the logarithmic term in his elementary solution is the Riemann-Green function of the adjoint equation. It is possible to modify HADAMARD'S construction so as to give both functions at the same time.

(*iii*) It is easy to construct an integral equation whose unique solution is the Riemann-Green function.

(*iv*) CHAUNDY, in his work on partial differential equations of hypergeometric type, was able to construct the Riemann-Green function by the use of symbolic operators and power series. This work appears to be little known.

(*v*) A. G. MACKIE has constructed complex integral solutions of certain equations. Such a complex integral gives the Riemann-Green function for an appropriate choice of contour. To some extent, MACKIE was anticipated by CHAUNDY, whose approach was rather different.

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* HADAMARD: Lectures on Cauchy's Problem in Linear Partial Differential Equations, p. 72. Dover Press 1952.

(*vi*) TITCHMARSH gave a direct solution of the characteristic boundary value problem defining V for the equation of damped waves by means of a complex Fourier integral.

§ 3. The method used by Riemann

Let us consider the Problem of Cauchy for the equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + 2a\frac{\partial U}{\partial x} - 2b\frac{\partial U}{\partial y} + cU = 0 \tag{3.1}$$

under the conditions

$$U = 0, \qquad \frac{\partial U}{\partial y} = F(x) \tag{3.2}$$

when $y = y_0$, where y_0 is an arbitrary constant. The solution (2.2) then reduces to*

$$U(X, Y) = \tfrac{1}{2}\int\limits_{X-Y+y_0}^{X+Y-y_0} V(x, y_0; X, Y)\,F(x)\,dx. \tag{3.3}$$

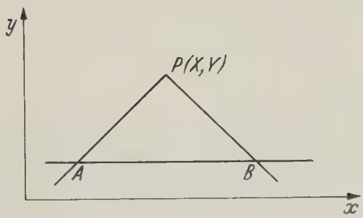

Fig. 3. Diagram for Eq. (3.3)

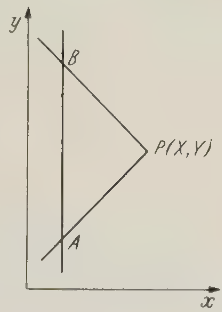

Fig. 4. Diagram for Eq. (3.5)

(See Fig. 3.) If we could solve this problem by some other method, a comparison of the two solutions would give $V(x, y_0; X, Y)$ when x lies between $X \pm (Y - y_0)$; as y_0 is arbitrary, this would give $V(x, y; X, Y)$ whenever $X - x$ lies between $\pm(Y - y)$. Similarly if the data were

$$U = 0, \qquad \frac{\partial U}{\partial x} = G(y) \tag{3.4}$$

when $x = x_0$, where x_0 is arbitrary, we should have

$$U(X, Y) = \tfrac{1}{2}\int\limits_{Y-X+x_0}^{Y+X-x_0} V(x_0, y; X, Y)\,G(y)\,dy. \tag{3.5}$$

(See Fig. 4.) If another form of the solution were available, we should get in this way $V(x, y; X, Y)$ when $Y - y$ lies between $\pm(X - x)$. The reasons for this are (*a*) the solution of the Problem of Cauchy is unique, and (*b*) the Riemann-Green function does not depend on the nature of the curve carrying the Cauchy data.

This is precisely what RIEMANN did in the special problem which interested him. The equation with which he was concerned was

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + \frac{2\alpha}{x}\frac{\partial U}{\partial x} = 0 \tag{3.6}$$

* Note that, if $Y < y_0$, the upper limit in this integral is less than the lower limit. Similarly in (3.5) if $X < x_0$. This is of importance in what follows.

in our notation, where α is a constant. He solved the problem of Cauchy under the conditions

$$U = 0, \qquad \frac{\partial U}{\partial x} = G(y)$$

when $x - x_0 > 0$ by means of a Fourier cosine transform with respect to y, and, by a comparison of the two forms of the solution, he obtained, in modern notation,

$$\begin{aligned} V(x, y; X, Y) = \frac{x^{\alpha+\frac{1}{2}}}{\cos\alpha\pi\, X^{\alpha-\frac{1}{2}}} \int_0^\infty \cos\left[\lambda(Y-y)\right] \times \\ \times \left[J_{\alpha-\frac{1}{2}}(\lambda x)\, J_{\frac{1}{2}-\alpha}(\lambda X) - J_{\alpha-\frac{1}{2}}(\lambda X)\, J_{\frac{1}{2}-\alpha}(\lambda x)\right] d\lambda \end{aligned} \tag{3.7}$$

when $Y - y$ lies between $\pm(X - x)$. He then remarked that each Bessel function can be replaced by a definite integral, so that V is then expressed as a triple integral, which can be reduced to a hypergeometric function. He merely quoted the final result, which nowadays we should write as

$$V(x, y; X, Y) = \left(\frac{x}{X}\right)^{\alpha} P_{-\alpha}(1 + \xi) \tag{3.8}$$

where

$$\xi = \frac{(X-x)^2 - (Y-y)^2}{2xX},$$

and contented himself with verifying the result. WEBER's editorial remarks in the "Collected Works" shed no light on how RIEMANN reached this remarkable result which antedated by twenty years the evaluation of similar integrals by SONINE*.

RIEMANN was able to obtain a second solution of his equation by means of a Fourier cosine transform because the variables are separable. We show in the next section that the method can be extended to any equation with separable variables.

§ 4. The generalisation of Riemann's method

When the variables are separable, equation (2.1) can be written in the form

$$\frac{\partial^2 U}{\partial x^2} + 2a\frac{\partial U}{\partial x} + pU = \frac{\partial^2 U}{\partial y^2} + 2b\frac{\partial U}{\partial y} + qU \tag{4.1}$$

where a, p are functions of x alone, b, q are functions of y alone. We consider, as is usual in such a case, the pair of equations

$$\frac{d^2\vartheta}{dx^2} + 2a\frac{d\vartheta}{dx} + (p + \lambda^2)\vartheta = 0 \tag{4.2}$$

$$\frac{d^2\varphi}{dy^2} + 2b\frac{d\varphi}{dy} + (q + \lambda^2)\varphi = 0 \tag{4.3}$$

where λ^2 is the separation constant. Suppose that $\vartheta_1(x, \lambda)$, $\vartheta_2(x, \lambda)$ are linearly independent solutions of (4.2) whose Wronskian

$$\vartheta_1\frac{d\vartheta_2}{dx} - \vartheta_2\frac{d\vartheta_1}{dx}$$

* SONINE: Math. Ann. **16**, 46 (1880). See also WATSON, G. N.: Theory of Bessel Functions, pp. 411—412. Cambridge 1922.

we denote by $W'(x, \lambda)$. Similarly $\varphi_1(y, \lambda)$, $\varphi_2(y, \lambda)$ are linearly independent solutions of (4.3) whose Wronskian is $W''(y, \lambda)$.

We now try to construct a solution of the problem of Cauchy for equation (4.1) of the form

$$U(x, y) = \int \{f_1(\lambda)\, \varphi_1(y, \lambda) + f_2(\lambda)\, \varphi_2(y, \lambda)\}\, \vartheta_1(x, \lambda)\, d\lambda$$

where integration is over a fixed range which depends on the form of the equation. We wish to find $U(X, Y)$, given that $U = 0$, $\partial U/\partial y = F(x)$ on $y = y_0$, where y_0 is a fixed but arbitrary constant. This means that we have to find $f_1(\lambda)$, $f_2(\lambda)$, if possible, so that

$$\int \{f_1(\lambda)\, \varphi_1(y_0, \lambda) + f_2(\lambda)\, \varphi_2(y_0, \lambda)\}\, \vartheta_1(x, \lambda)\, d\lambda = 0$$

$$\int \{f_1(\lambda)\, \dot\varphi_1(y_0, \lambda) + f_2(\lambda)\, \dot\varphi_2(y_0, \lambda)\}\, \vartheta_1(x, \lambda)\, d\lambda = F(x),$$

where $\dot\varphi_i(y, \lambda)$ denotes $d\varphi_i/dy$. If

$$F(x) = \int f(\lambda)\, \vartheta_1(x, \lambda)\, d\lambda, \tag{4.4}$$

these conditions are satisfied by

$$f_1(\lambda)\, \varphi_1(y_0, \lambda) + f_2(\lambda)\, \varphi_2(y_0, \lambda) = 0$$

$$f_1(\lambda)\, \dot\varphi_1(y_0, \lambda) + f_2(\lambda)\, \dot\varphi_2(y_0, \lambda) = f(\lambda),$$

that is, by

$$f_1(\lambda) = -\frac{f(\lambda)\, \varphi_2(y_0, \lambda)}{W''(y_0, \lambda)}, \qquad f_2(\lambda) = \frac{f(\lambda)\, \varphi_1(y_0, \lambda)}{w''(y_0, \lambda)}.$$

Hence we get

$$U(X, Y) = \int f(\lambda)\, \{\varphi_1(y_0, \lambda)\, \varphi_2(Y, \lambda) - \varphi_1(Y, \lambda)\, \varphi_2(y_0, \lambda)\}\, \frac{\vartheta_1(X, \lambda)}{W''(y_0, \lambda)}\, d\lambda.$$

But if the solution of the integral equation (4.4) is

$$f(\lambda) = \int \bar\vartheta_1(x, \lambda)\, F(x)\, dx, \tag{4.5}$$

we then have

$$U(X, Y) = \iint f(x)\, \frac{\vartheta_1(X, \lambda)\, \bar\vartheta_1(x, \lambda)}{W''(y_0, \lambda)}\, \{\varphi_1(y_0, \lambda)\, \varphi_2(Y, \lambda) - \varphi_1(Y, \lambda)\, \varphi_2(y_0, \lambda)\}\, dx\, d\lambda.$$

Comparing this with

$$U(X, Y) = \tfrac{1}{2} \int_{X-Y+y_0}^{X+Y-y_0} f(x)\, V(x, y_0; X, Y)\, dx,$$

we see that

$$V(x, y; X, Y) = \pm 2 \int \frac{\vartheta_1(X, \lambda)\, \bar\vartheta_1(x, \lambda)}{W''(y, \lambda)}\, \{\varphi_1(y, \lambda)\, \varphi_2(Y, \lambda) - \varphi_1(Y, \lambda)\, \varphi_2(y, \lambda)\}\, d\lambda \tag{4.6}$$

provided that x lies between $X \pm (Y - y)$, but that, when x is outside this range, the value of the integral is zero; the upper sign is taken if $Y > y$, the lower if $Y < y$.

Interchanging the roles of x and y, we see also that

$$V(x, y; X, Y) = \pm 2 \int \frac{\varphi_1(Y, \lambda)\, \bar\varphi_1(y, \lambda)}{W''(x, \lambda)}\, \{\vartheta_1(x, \lambda)\, \vartheta_2(X, \lambda) - \vartheta_1(X, \lambda)\, \vartheta_2(x, \lambda)\}\, d\lambda \tag{4.7}$$

if the solution of the integral equation

$$\int g(\lambda)\, \varphi_1(y, \lambda) = G(y) \tag{4.8}$$

is

$$g(\lambda) = \int \overline{\varphi}_1(y, \lambda)\, G(y)\, dy, \tag{4.9}$$

provided that y lies between $Y \pm (X - x)$, but that, when y is outside this range, the values of the integral is zero; the upper sign is taken if $X > x$, the lower if $X < x$.

Lastly, we observe that two other formulae for V can be obtained by interchanging the suffixes 1 and 2 and changing the sign of W' (or W''). This comes to replacing ϑ_1 and $\overline{\vartheta}_1$ in (4.6) by ϑ_2 and the corresponding $\overline{\vartheta}_2$; and similarly in (4.7).

All this is entirely formal. To give a rigorous discussion would involve a careful consideration of the various cases which can arise, and the essential idea would be lost in a mass of detail. The important point is that, in many special problems, this technique gives the form of the Riemann-Green function; having got the form, one can usually justify the result quite easily.

§ 5. Some particular examples

As a first example, we consider the equation of damped waves

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + U = 0 \tag{5.1}$$

The two associated equations are

$$\frac{d^2 \vartheta}{dx^2} + (\lambda^2 + 1)\, \vartheta = 0,$$

$$\frac{d^2 \varphi}{dy^2} + \lambda^2 \varphi = 0.$$

Hence

$$\vartheta_1 = e^{-ix\sqrt{\lambda^2+1}}, \qquad \vartheta_2 = e^{ix\sqrt{\lambda^2+1}}, \qquad W' = 2i\sqrt{(\lambda^2 + 1)},$$

and

$$\varphi_1 = e^{-i\lambda y}, \qquad \varphi_2 = e^{i\lambda y}, \qquad W'' = 2i\lambda.$$

If we use formula (4.7), we are in fact solving the equation by means of complex Fourier transforms in y. For the range is $-\infty < y < \infty$, and $\overline{\varphi}_1$ is defined by the fact that the solution of

$$\int_{-\infty}^{\infty} g(\lambda)\, e^{-i\lambda y}\, d\lambda = G(y)$$

is

$$g(\lambda) = \int_{-\infty}^{\infty} G(y)\, \overline{\varphi}_1(y, \lambda)\, dy.$$

Hence

$$\overline{\varphi}_1(y, \lambda) = \frac{1}{2\pi}\, e^{i\lambda y}.$$

It follows from our formal analysis that

$$\pm \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\lambda(Y-y)}\, \frac{\sin\{\sqrt{\lambda^2+1}\,(X-x)\}}{\sqrt{(\lambda^2+1)}}\, d\lambda = V(x, y; X, Y)$$

provided that y lies between $Y \pm (X - x)$, and that otherwise the value of the integral is zero. We may evidently suppose $X > x$ and take the upper sign; and since the integral can be written as

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \cos \lambda (Y - y) \frac{\sin\{\sqrt{\lambda^2 + 1}\,(X - x)\}}{\sqrt{(\lambda^2 + 1)}} \, d\lambda,$$

we may also suppose $Y > y$. To evaluate the integral when y lies between $Y \pm (X - x)$, we put

$$X - x = R \cosh \alpha, \qquad Y - y = R \sinh \alpha, \qquad \lambda = \sinh \vartheta$$

and get

$$V = \frac{1}{2\pi i} \int_{\infty}^{\infty} \left\{ e^{i R \cosh(\vartheta - \alpha)} - e^{-i R \cosh(\vartheta + \alpha)} \right\} d\vartheta = \frac{1}{2} \left\{ H_0^{(1)}(R) + H_0^{(2)}(R) \right\} = J_0(R)$$

or

$$V(x, y; X, Y) = J_0 \left[\sqrt{\{(X - x)^2 - (Y - y)^2\}} \right]. \tag{5.2}$$

When $Y - y$ does not lie between $\pm (X - x)$, we may evidently consider only the case $Y - y > X - x$: if we make the substitutions

$$X - x = R \sinh \alpha, \qquad Y - y = R \cosh \alpha, \qquad \lambda = \sinh \vartheta$$

we find that the integral vanishes. This does not mean that the Riemann-Green function vanishes in this case, but only that to find it we must have recourse to formula (4.6).

When $X - x$ lies between $\pm (Y - y)$, we use $\lambda^2 - 1$ as separation constant instead of λ^2, and use Fourier transforms in x. The result is that in this case

$$\pm \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\lambda(X - x)} \frac{\sin\{\sqrt{\lambda^2 - 1}\,(Y - y)\}}{\sqrt{(\lambda^2 - 1)}} \, d\lambda = V(x, y; X, Y)$$

where the upper or lower sign is taken as $Y >$ or $< y$, and that otherwise the integral is zero. It turns out that

$$V(x, y; X, Y) = I_0 \left[\sqrt{\{(Y - y)^2 - (X - x)^2\}} \right]. \tag{5.3}$$

The equation (5.3) appears, very slightly disguised, in RIEMANN's paper as the limiting form of the Riemann-Green function for the equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial y^2} + \frac{2\alpha}{x} \frac{\partial U}{\partial x} = 0 \tag{5.4}$$

where α is a constant*.

Since (5.4) has a singular line $x = 0$, we restrict consideration to one of the half-planes, say $x \geqq 0$, into which the plane is divided by the singular line: in what follows, x and X are positive. The associated equations are

$$\frac{d^2 \vartheta}{dx^2} + \frac{2\alpha}{x} \frac{d\vartheta}{dx} + \lambda^2 \vartheta = 0,$$

$$\frac{d^2 \varphi}{dy^2} + \lambda^2 \varphi = 0,$$

* Put $x = \alpha + t$ and make α tend to infinity. Then put $U = e^{-t} U_1$.

so that we may take

$$\vartheta_1 = x^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda x), \quad \vartheta_2 = x^{\frac{1}{2}-\alpha} J_{\frac{1}{2}-\alpha}(\lambda x), \quad W' = \frac{2\cos\pi\alpha}{x^{2\alpha}},$$

$$\varphi_1 = e^{-i\lambda y}, \quad \varphi_2 = e^{i\lambda y}, \quad W'' = 2i\lambda,$$

if $\alpha - \frac{1}{2}$ is not an integer, as we suppose to be the case. If $\alpha - \frac{1}{2}$ is an integer, we must take

$$\vartheta_1 = x^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda x), \quad \vartheta_2 = x^{\frac{1}{2}-\alpha} y_{\alpha-\frac{1}{2}}(\lambda x), \quad W' = \frac{2}{x^{2\alpha}};$$

but the final result will be the same. If we use (4.7), which means that we are solving (5.4) by complex Fourier transforms in y, then y and λ vary from $-\infty$ to $+\infty$ and $\bar{\varphi}_1 = e^{i\lambda y}/2\pi$. We then have, when $y - Y$ lies between $\pm(x - X)$,

$$\begin{aligned} V(x, y; X, Y) &= \frac{\pm x^{\alpha+\frac{1}{2}}}{2\cos\pi\alpha\, X^{\alpha-\frac{1}{2}}} \int_{-\infty}^{\infty} e^{i\lambda(y-Y)} \{J_{\alpha-\frac{1}{2}}(\lambda x) J_{\frac{1}{2}-\alpha}(\lambda X) - J_{\alpha-\frac{1}{2}}(\lambda X) J_{\frac{1}{2}-\alpha}(\lambda x)\}\, d\lambda \\ &= \frac{\pm x^{\alpha+\frac{1}{2}}}{\cos\pi\alpha\, X^{\alpha-\frac{1}{2}}} \int_0^{\infty} \cos\lambda(y-Y) \{J_{\alpha-\frac{1}{2}}(\lambda x) J_{\frac{1}{2}-\alpha}(\lambda X) - J_{\alpha-\frac{1}{2}}(\lambda X) J_{\frac{1}{2}-\alpha}(\lambda x)\}\, d\lambda \end{aligned} \tag{5.5}$$

since the expression in brackets is an even function of λ; for other values of $y - Y$, the integral must be zero. This is the expression given by Riemann. The plus or minus sign is taken according as $X >$ or $< x$.

To evaluate this integral, we use the result* that, if $a \geqq b > 0$, $c > 0$,

$$\begin{aligned} \int_0^{\infty} \cos c\lambda\, J_\nu(b\lambda)\, J_{-\nu}(a\lambda)\, d\lambda &= \frac{\cos\nu\pi}{\pi\sqrt{(ab)}}\, Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2-c^2}{2ab}\right) \quad (0 < c < a-b) \\ &= \frac{1}{2\sqrt{(ab)}}\, P_{\nu-\frac{1}{2}}\left(\frac{c^2-a^2-b^2}{2ab}\right) \quad (a-b < c < a+b) \\ &= 0 \quad (c > a+b). \end{aligned} \tag{5.6}$$

We may evidently suppose that $y > Y$, $x < X$; but we must also have regard to the position of (x, y) in relation to the characteristics through (X, Y) and its image $(-X, Y)$ in the singular line. These characteristics divide the half-plane into six regions as shown in Fig. 5.

In *III*, $y - Y > x + X$, and so the integral vanishes. In the part of *II* where $x < X$, we have $X - x < y - Y < X + x$ and therefore

$$\begin{aligned} &\frac{x^{\alpha+\frac{1}{2}}}{\cos\pi\alpha\, X^{\alpha-\frac{1}{2}}} \int_0^{\infty} \cos\lambda(y-Y) \{J_{\alpha-\frac{1}{2}}(\lambda x) J_{\frac{1}{2}-\alpha}(\lambda X) - J_{\alpha-\frac{1}{2}}(\lambda X) J_{\frac{1}{2}-\alpha}(\lambda x)\}\, d\lambda \\ &= \frac{1}{2\cos\pi\alpha} \frac{x^\alpha}{X^\alpha} \left\{P_{\alpha-1}\left(\frac{(y-Y)^2-x^2-X^2}{2xX}\right) - P_{-\alpha}\left(\frac{(y-Y)^2-x^2-X^2}{2xX}\right)\right\} = 0. \end{aligned}$$

* The only reference I can give is to Oberhettinger, F.: Tabellen zur Fourier Transformation, p. 65. Berlin: Springer 1957. That the integral vanishes when $c > a + b$ was proved by Bailey: Proc. London Math. Soc. (2) **40**, 37–48 (1936).

Lastly, in the part of I where $y-Y>0$, we have $0<x<X$, $0<y-Y<X-x$, and so

$$V(x, y; X, Y) = \frac{\tan \pi\alpha}{\pi} \frac{x^\alpha}{X^\alpha} \left\{ Q_{\alpha-1}\left(\frac{x^2+X^2-(y-Y)^2}{2xX}\right) - Q_{-\alpha}\left(\frac{x^2+X^2-(y-Y)^2}{2xX}\right) \right\}$$

or

$$V(x, y; X, Y) = \frac{x^\alpha}{X^\alpha} P_{-\alpha}\left(\frac{x^2+X^2-(y-Y)^2}{2xX}\right). \tag{5.7}$$

It is now evident that the integral (5.5) vanishes except in the regions I and I', and that, in these regions, the Riemann-Green function is given by (5.7).

To get the Riemann-Green function in the rest of the half-plane, we use the formula (4.6). This means that we are solving the partial differential equation by means of a Hankel transform in x. The function $\bar{\vartheta}_1(x, \lambda)$ is to be chosen so that the solution of

$$F(x) = \int_0^\infty x^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda x) f(\lambda)\, d\lambda$$

is

$$f(\lambda) = \int_0^\infty \bar{\vartheta}_1(x, \lambda) F(x)\, dx.$$

It follows that, when $\alpha>0$

$$\bar{\vartheta}_1(x, \lambda) = \lambda x^{\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lambda x).$$

Hence we have the result that

$$\pm 2 \frac{x^{\alpha+\frac{1}{2}}}{X^{\alpha-\frac{1}{2}}} \int_0^\infty J_{\alpha-\frac{1}{2}}(\lambda x) J_{\alpha-\frac{1}{2}}(\lambda X) \sin \lambda(Y-y)\, d\lambda$$

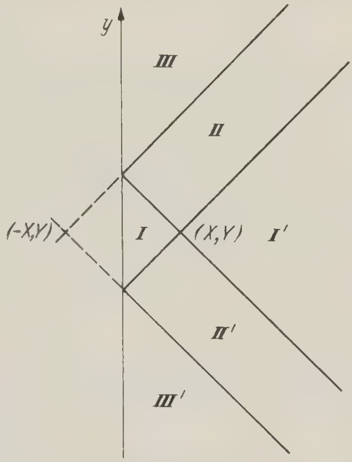

Fig. 5. Regions for calculation of V for Eq. (5.4)

is the required Riemann-Green function when $X-x$ lies between $\pm(Y-y)$, the upper or lower sign being taken according as $Y>$ or $<y$, and that otherwise the integral is zero.

To evaluate this integral, we use a result due to H. M. MACDONALD*, that if $a \geqq b>0$, $c>0$, $\nu>-\frac{1}{2}$,

$$\begin{aligned} \int_0^\infty \sin c\lambda\, J_\nu(b\lambda)\, J_\nu(a\lambda)\, d\lambda &= 0 && (0<c<a-b) \\ &= \frac{1}{2\sqrt{(ab)}} P_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2-c^2}{2ab}\right) && (a-b<c<a+b) \\ &= \frac{\cos \nu\pi}{\pi\sqrt{(ab)}} Q_{\nu-\frac{1}{2}}\left(\frac{c^2-a^2-b^2}{2ab}\right) && (c>a+b). \end{aligned} \tag{5.8}$$

As before, we have to consider the six regions into which the half plane is divided by the characteristics through (X, Y) and $(-X, Y)$; and we may evidently suppose that $Y-y$ is positive.

* *Cf.* WATSON: Theory of Bessel Functions, pp. 411—412. Cambridge 1922.

In the part of I for which $Y > y$, we have $0 < Y - y < X - x$; hence by the above formula with $c = Y - y$, $b = x$, $a = X$, the integral vanishes. Similarly in the part of I' for which $Y > y$, we have $0 < Y - y < x - X$, and the integral again vanishes by taking $c = Y - y$, $b = X$, $a = x$. Next in the part of II' for which $x < X$, we have $X - x < Y - y < X + x$, so that there

$$V(x, y; X, Y) = \frac{x^\alpha}{X^\alpha} P_{-\alpha}\left(\frac{x^2 + X^2 - (y - Y)^2}{2xX}\right) \tag{5.9}$$

since $P_{-\alpha} = P_{\alpha-1}$. And in the part of II' for which $x > X$, we have $x - X < Y - y < x + X$, so that the same formula holds. Lastly in III', $Y - y > X + x$ so that in this region

$$V(x, y; X, Y) = \frac{2}{\pi} \sin \alpha\pi \frac{x^\alpha}{X^\alpha} Q_{\alpha-1}\left(\frac{(y - Y)^2 - x^2 - X^2}{2xX}\right). \tag{5.10}$$

To sum up, the Riemann-Green function for equation (5.4) is given by (5.6) everywhere in the half-plane $x > 0$, except the regions III and III' where formula (5.10) holds. The unusual behaviour of V in the regions III and III' does not appear to have been noted before: yet it would be of importance if one tried to use Riemann's method with data on the axes of coordinates and wished to find the solution when $Y > X$. If α is an integer, V is identically zero in III and III'.

As a last example, let us consider the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{2\alpha}{x} \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y} \frac{\partial U}{\partial y} \tag{5.11}$$

where α, β are constants. The associated equations are

$$\frac{d^2\vartheta}{dx^2} + \frac{2\alpha}{x} \frac{d\vartheta}{dx} + \lambda^2 \vartheta = 0$$

$$\frac{d^2\varphi}{dy^2} + \frac{2\beta}{y} \frac{d\varphi}{dy} + \lambda^2 \varphi = 0,$$

so that we may take

$$\vartheta_1 = x^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(\lambda x), \quad \vartheta_2 = x^{\frac{1}{2}-\alpha} J_{\frac{1}{2}-\alpha}(\lambda x), \quad W' = \frac{2\cos\pi\alpha}{\pi x^{2\alpha}},$$

$$\varphi_1 = y^{\frac{1}{2}-\beta} J_{\beta-\frac{1}{2}}(\lambda y), \quad \varphi_2 = y^{\frac{1}{2}-\beta} J_{\frac{1}{2}-\beta}(\lambda y), \quad W'' = \frac{2\cos\pi\beta}{\pi y^{2\beta}}.$$

As there is symmetry in (x, y), we may use (4.6) or (4.7). If we use (4.6), we can deduce the formula corresponding to (4.7) by interchanging x and y and α and β.

We take x, y, X, Y to be positive, which corresponds to solving the partial differential equation in the first quadrant of the (x, y) plane. As in the previous example,

$$\bar{\vartheta}_1(x, \lambda) = \lambda x^{\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lambda x)$$

and so, if $\beta - \frac{1}{2}$ is not an integer, the integral we have to consider is

$$\pm \frac{\pi}{\cos\pi\beta} \frac{x^{\alpha+\frac{1}{2}} y^{\beta+\frac{1}{2}}}{X^{\alpha-\frac{1}{2}} Y^{\beta-\frac{1}{2}}} \int_0^\infty \lambda J_{\alpha-\frac{1}{2}}(\lambda x) J_{\alpha-\frac{1}{2}}(\lambda X) \times$$

$$\times \{J_{\beta-\frac{1}{2}}(\lambda y) J_{\frac{1}{2}-\beta}(\lambda Y) - J_{\beta-\frac{1}{2}}(\lambda Y) J_{\frac{1}{2}-\beta}(\lambda y)\} \, d\lambda \tag{5.12}$$

which should give $V(x, y; X, Y)$ when $X-x$ lies between $\pm(Y-y)$, but should be zero otherwise; the upper or lower sign is taken according as $Y >$ or $< y$. The final result will, however, hold if $\beta-\frac{1}{2}$ is an integer.

To evaluate this integral, we use the following result in the theory of Fourier transforms, that if

$$F(u)=\int_0^\infty f(\lambda)\cos\lambda u\,d\lambda$$

$$G(u)=\int_0^\infty g(\lambda)\sin\lambda u\,d\lambda,$$

then

$$\int_0^\infty \lambda f(\lambda)\,g(\lambda)\,d\lambda=\int_0^\infty F(u)\,dG(u),$$

the last integral being a Stieltjes integral, since $G(u)$ in our problem turns out to be discontinuous.

If we take

$$f(\lambda)=\pm\{J_{\beta-\frac{1}{2}}(\lambda y)\,J_{\frac{1}{2}-\beta}(\lambda Y)-J_{\frac{1}{2}-\beta}(\lambda y)\,J_{\beta-\frac{1}{2}}(\lambda Y)\},$$

where the upper or lower sign is taken according as $Y >$ or $< y$, it follows from (5.6) that

$$\begin{aligned} F(u) &= \sqrt{\frac{2}{\pi y Y}}\cos\pi\beta\,P_{-\beta}\left(\frac{y^2+Y^2-u^2}{2yY}\right) && \text{if } 0<u<|Y-y|, \\ &= 0 && \text{if } u>|Y-y|. \end{aligned}$$

If we take

$$g(\lambda)=J_{\alpha-\frac{1}{2}}(\lambda x)\,J_{\alpha-\frac{1}{2}}(\lambda X),$$

it follows from (5.8) that if $\alpha>0$

$$\begin{aligned} G(u) &= 0 && \text{if } 0<u<|X-x| \\ &= \frac{1}{\sqrt{(2\pi x X)}}\,P_{\alpha-1}\left(\frac{x^2+X^2-u^2}{2xX}\right) && \text{if } |X-x|<u<X+x \\ &= \frac{\sin\alpha\pi}{\sqrt{(\frac{1}{2}\pi^3 x X)}}\,Q_{\alpha-1}\left(\frac{u^2-x^2-X^2}{2xX}\right) && \text{if } u>X+x, \end{aligned}$$

where $P_{\alpha-1}\equiv P_{-\alpha}$. It follows that

$$\int_0^\infty \lambda f(\lambda)\,g(\lambda)\,d\lambda=\int_0^{|Y-y|} F(u)\,dG(u).$$

We have to consider separately the cases $X>Y$ and $X<Y$. In Figs. 6 and 7, the quadrant $x\geqq 0$, $y\geqq 0$ is divided up into regions by characteristics $x\pm y=$ constant. In the regions I and I', $|Y-y|<|X-x|$ for both cases; $G(u)$ vanishes everywhere in the range of integration, and so the integral (5.12) vanishes there as our argument led us to expect. This, of course, does not mean that V is identically zero there, but only that our argument does not determine V.

Now let $X > Y$, so that the situation is as in Fig. 6. Then in II' we have $-(Y-y) < X-x < Y-y$. Hence

$$\int_0^{\infty} \lambda f(\lambda)\, g(\lambda)\, d\lambda = \left[F(u)\, G(u)\right]_{|X-x|-0}^{|X-x|+0} + \int_{|X-x|+0}^{Y-y} F(u)\, dG(u)$$

$$= \frac{\cos \pi\beta}{\pi \sqrt{(x X y Y)}} \left[P_{-\beta}\left(\frac{y^2+Y^2-(X-x)^2}{2yY}\right) + \int_{|X-x|}^{Y-y} P_{-\beta}\left(\frac{y^2+Y^2-u^2}{2yY}\right) dP_{-\alpha}\left(\frac{x^2+X^2-u^2}{2xX}\right)\right]$$

$$= \frac{\cos \pi\beta}{\pi \sqrt{(x X y Y)}} \left[P_{-\beta}(1+\eta) + \int_0^{\pi/2} P_{-\beta}(1+\eta\cos^2\vartheta)\, dP_{-\alpha}(1+\xi\sin^2\vartheta)\right]$$

where

$$\xi = \frac{(X-x)^2-(Y-y)^2}{2xX},$$

$$\eta = \frac{(Y-y)^2-(X-x)^2}{2yY};$$

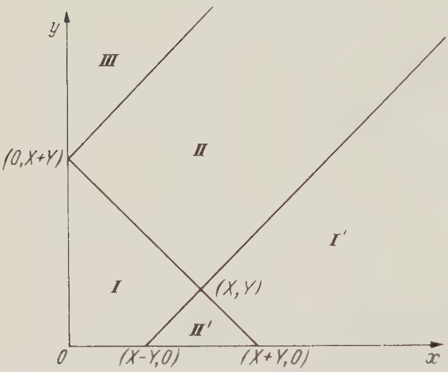

Fig. 6. Regions for calculation of V for Eq. (5.11) ($X > Y$)

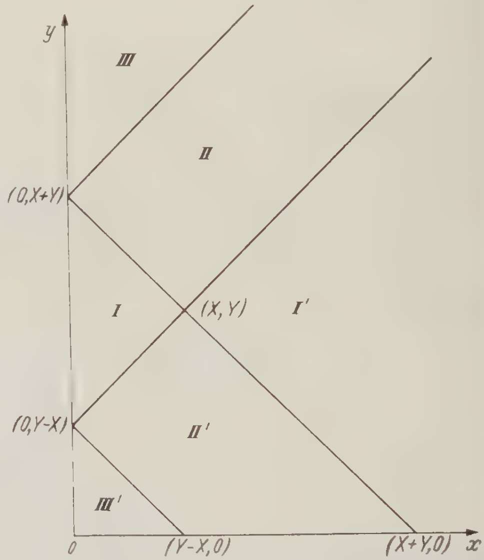

Fig. 7. Regions for calculation of V for Eq. (5.11) ($Y > X$)

this follows by making the substitution

$$u^2 = (X-x)^2 \cos^2\vartheta + (Y-y)^2 \sin^2\vartheta.$$

We thus have the result that, if $X > Y$,

$$V(x, y; X, Y) = \frac{x^{\alpha} y^{\beta}}{X^{\alpha} Y^{\beta}} \left[P_{-\beta}(1+\eta) + \int_0^{\pi/2} P_{-\beta}(1+\eta\cos^2\vartheta)\, dP_{-\alpha}(1+\xi\sin^2\vartheta)\right] \quad (5.13)$$

in II'. Since ξ and η vanish on the characteristics through (X, Y), this function obviously satisfies the prescribed boundary conditions. When I discovered this result in the summer of 1957, I thought it was new; but I have since then learned that CHAUNDY* gave it nearly twenty years ago in a paper which does not appear to be well known; his derivation is completely different from mine. (See § 8.) Similarly it can be shown that the formula (5.13) holds in II.

* CHAUNDY, T. W.: Quarterly Journ. of Math., Oxford Ser. **2**, 234–240 (1938).

The situation is rather more difficult in region III, for there we have

$$|x-X| < x+X < y-Y,$$

so that

$$V(x,y;X,Y) = \left[F(u)\,G(u)\right]_{|X-x|-0}^{|X-x|+0} + \int_{|X-x|+0}^{X+x-0} F(u)\,dG(u) + \\ + \left[F(u)\,G(u)\right]_{X+x-0}^{X+x+0} + \int_{X+x+0}^{y-Y} F(u)\,dG(u).$$

The difficulty is that $G(u)$ tends to infinite limits as $u \to X+x\pm 0$, which makes the resulting formula more complicated.

If, however, $Y > X$, as in Fig. 7, everything goes as before: the Riemann-Green function is given by (5.13) in II and II', and more complicated formulae hold in III and III'.

Lastly, to get V in I and I', we interchange the roles played by x and y, and by α and β, and make the corresponding changes in Figs. 6 and 7.

A different form of the Riemann-Green function for (5.11) has recently been given by P. HENRICI*, *viz.*

$$\left(\frac{x}{X}\right)^{\alpha}\left(\frac{y}{Y}\right)^{\beta} F_3\left(\alpha,\beta,1-\alpha,1-\beta;\,1;\,-\frac{R^2}{4xX},\,\frac{R^2}{4yY}\right) \tag{5.14}$$

where

$$R^2 = (x-X)^2 - (y-Y)^2$$

and F_3 denotes APPELL'S hypergeometric function**.

As a last example, I would like to mention briefly the equation

$$\frac{\partial^2 U}{\partial x^2} - \varkappa^2 U = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y}\frac{\partial U}{\partial y} \tag{5.15}$$

for which HENRICI (*loc. cit.*) has given the Riemann-Green function

$$V(x,y;X,Y) = \frac{y^{\beta}}{Y^{\beta}}\,\Xi_2\left(\beta,1-\beta;1;\frac{R^2}{4yY},\,\frac{1}{4}\varkappa R^2\right), \tag{5.16}$$

This equation has its variables separated and could be discussed by the methods of this section. Another method would be to apply the limiting process suggested by RIEMANN*** to equation (5.11). If we put $x = (\alpha + \varkappa t)/\varkappa$ and make $\alpha \to \infty$, (5.11) becomes

$$\frac{\partial^2 U}{\partial t^2} + 2\varkappa\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y}\frac{\partial U}{\partial y}, \tag{5.17}$$

and the expression (5.13) for the Riemann-Green function then reduces to

$$V(t,y;T,X) = e^{\varkappa(t-T)}\frac{y^{\beta}}{Y^{\beta}} \times \\ \times\left[P_{-\beta}(1+\eta') + \int_0^{\pi/2} P_{-\beta}(1+\eta'\cos^2\vartheta)\,dJ_0\left(\varkappa\sin\vartheta\sqrt{\{(Y-y)^2-(T-t)^2\}}\right)\right] \tag{5.18}$$

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* HENRICI, P.: Z.A.M.P. **8**, 169—203 (1957); in particular Table 2 on p. 180.
** See WHITTAKER & WATSON: Modern Analysis, p. 300, Ex. 22. Cambridge 1920.
*** See the footnote on p. 331.

when $T-t$ lies between $\pm(Y-y)$, and

$$\eta' = \frac{(Y-y)^2-(T-t)^2}{2yY}.$$

It readily follows that the Riemann-Green function for

$$\frac{\partial^2 U}{\partial t^2} - \varkappa^2 U = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y}\frac{\partial U}{\partial y}$$

is obtained by omitting the exponential term in (5.18).

§ 6. Connexion with the work of Hadamard and Marcel Riesz

Riesz's theory of integrals of fractional order was originally devised to give a solution of the wave equation in any number of dimensions and was subsequently extended by him to deal with the equation $\Delta_2 U = 0$ in a Riemannian space of normal hyperbolic type. The extension to a non-self-adjoint equation, the Euler-Poisson-Darboux equation, was worked out in detail by Ruth M. Davis*.

In the case of two independent variables, say for the equation

$$LU \equiv \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + 2a(x,y)\frac{\partial U}{\partial x} - 2b(x,y)\frac{\partial U}{\partial y} + c(x,y)\,U = 0,$$

this work depends on finding a function $V^\alpha(x,y)$, depending on a parameter α, such that

$$L^* V^{\alpha+2} = V^\alpha,$$

(L^* being the operator adjoint to L) and also such that, if α is large enough, V^α vanishes on the characteristics through a general point (X, Y). This function V^α is closely related to Hadamard's elementary solution. It has the form**

$$V^{2\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma^{\alpha+j-1}\, V_j}{2^{2\alpha+j-1}\,\Gamma(\alpha)\,\Gamma(\alpha+j)}$$

where

$$\Gamma = (x-X)^2 - (y-Y)^2$$

and V_j is independent of α.

In general, $V^{2\alpha}$ has a simple zero at $\alpha = 0$. The equation

$$L^* V^{2\alpha+2} = V^{2\alpha}$$

then implies that $L^* V^2 = 0$; and V^2 turns out to be the required Riemann-Green function. If all the V_j were zero, $V^{2\alpha}$ would have a double zero at $\alpha = 0$, and in this case $[\partial V^{2\alpha}/\partial\alpha]_{\alpha=0}$ would also be a solution, actually a constant multiple of Hadamard's elementary solution. This, however, only happens if a, b, c are all identically zero.

* Davis, Ruth M.: Annali di Mat. (IV) **42**, 205—226 (1956).

** It is simpler to consider $V^{2\alpha}$ rather than V^α: it avoids annoying fractions $\frac{1}{2}\alpha$ in the denominator.

It is, however, possible to proceed in a slightly different way, analogous to the method of FROBENIUS for ordinary linear equations. Let us consider, for example, the function

$$W^{2\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma^{\alpha+j-1}}{2^{2j}\,\Gamma(\alpha+j)\,\Gamma(\alpha+j)}.$$

It is readily shown that

$$\frac{\partial^2 W^{2\alpha+2}}{\partial x^2} - \frac{\partial^2 W^{2\alpha+2}}{\partial y^2} - W^{2\alpha+2} = \frac{4\alpha^2\,\Gamma^{\alpha-1}}{\Gamma(\alpha+1)\,\Gamma(\alpha+1)}.$$

Since the right-hand side has a double zero when $\alpha = 0$, the functions W^2 and $[\partial W^{2\alpha+2}/\partial\alpha]_{\alpha=0}$ are solutions of

$$\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} - W = 0.$$

The former is the Riemann-Green function $I_0(\sqrt{\Gamma})$, the latter an elementary solution of the form

$$I_0(\sqrt{\Gamma})\log\Gamma - \sum_{j=0}^{\infty} \frac{\Gamma^j\,\psi(j+1)}{2^{2j-1}\,j!\,j!},$$

$\psi(t)$ being the logarithmic derivative of $\Gamma(t)$. We note that the Riemann-Green function is the coefficient of $\log\Gamma$ in the elementary solution.

The difference between the procedure we suggest here and that used by Miss DAVIS, is that we do not require V_j to be independent of α. We consider instead a function

$$W^{2\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma^{\alpha+j-1}\,W_j^{2\alpha}}{\Gamma(\alpha+j)\,\Gamma(\alpha+j)\,2^{2j}}.$$

It easily follows that

$$L^* W^{2\alpha+2} = \frac{4\alpha^2\,\Gamma^{\alpha-1}\,W_0^{2\alpha+2}}{\Gamma(\alpha+1)\,\Gamma(\alpha+1)}$$

provided that

$$s\,\frac{dW_0^{2\alpha+2}}{ds} - [(x-X)\,a + (y-Y)\,b]\,W_0^{2\alpha+2} = 0$$

$$s\,\frac{dW_j^{2\alpha+2}}{ds} + (\alpha+j)\,W_j^{2\alpha+2} - [(x-X)\,a + (y-Y)\,b]\,W_j^{2\alpha+2} + (\alpha+j)\,L^* W_{j-1}^{2\alpha+2} = 0,$$

where

$$s\,\frac{dW}{ds} = (x-X)\frac{\partial W}{\partial x} + (y-Y)\frac{\partial W}{\partial y}.$$

If we require $W_0^{2\alpha+2}$ to have the value unity at (X, Y), we find that

$$W_0^{2\alpha+2} = \exp\int_{(X,Y)}^{(x,y)} \{(x-X)\,a + (y-Y)\,b\}\,\frac{ds}{s},$$

integration being along the straight line from (X, Y) to (x, y). Hence if (X, Y) is not a singularity of either coefficient a or b, $W_0^{2\alpha+2}$ is regular in a neighbourhood of (X, Y) and is independent of α. We denote it simply by W_0. Then since

$$L^* W^{2\alpha+2} - \frac{4\alpha^2\,\Gamma^{\alpha-1}\,W_0}{\Gamma(\alpha+1)\,\Gamma(\alpha+1)},$$

it follows that W^2 and $[\partial W^{2\alpha+2}/\partial\alpha]_{\alpha=0}$ are solutions of $L^* W = 0$. The former will be the Riemann-Green function, the latter the elementary solution; the Riemann-Green function is the coefficient of $\log \Gamma$ in the elementary solution.

We are, however, somewhat anticipating the result. We must first show that the remaining coefficients $W_j^{2\alpha+2}$ can be determined uniquely, that the resulting series for $W^{2\alpha+2}$ is convergent and can be differentiated term by term.

Since,

$$a(x-X)+b(y-Y)=\frac{s}{W_0}\frac{dW_0}{ds},$$

the equation for $W_j^{2\alpha+2}$ can be written as

$$\frac{d}{ds}\left[s^{\alpha+j}\frac{W_j^{2\alpha+2}}{W_0}\right]+(\alpha+j)\,s^{\alpha+j-1}\frac{L^* W_{j-1}^{2\alpha+2}}{W_0}=0,$$

and so

$$W_j^{2\alpha+2}=\frac{A_j}{s^{\alpha+j}}W_0-\frac{(\alpha+j)\,W_0}{s^{\alpha+j}}\int\limits_{(X,Y)}^{(x,y)}\frac{s^{\alpha+j-1}}{W_0}L^* W_{j-1}^{2\alpha+2}\,ds$$

where A_j is a constant of integration. If this is to be regular near (X, Y) no matter how large α is, A_j is zero for all j. Hence

$$W_0=\exp\int\limits_{(X,Y)}^{(x,y)}\{(x-X)\,a+(x-Y)\,b\}\frac{ds}{s},$$

$$W_j^{2\alpha+2}=-\frac{(\alpha+j)\,W_0}{s^{\alpha+j}}\int\limits_{(X,Y)}^{(x,y)}\frac{s^{\alpha+j-1}}{W_0}L^* W_{j-1}^{2\alpha+2}\,ds.$$

The convergence proof, when the coefficients are holomorphic, goes through as in Hadamard's book, and the rest follows. For

$$W^2=W_0+\sum_{j=1}^{\infty}\frac{\Gamma^j W_j^2}{j!\,j!\,2^{2j}}$$

gives

$$\frac{\partial W^2}{\partial x}=\frac{\partial W_0}{\partial x}+\frac{1}{2}(x-X)\,W_1^2+O(\Gamma),$$

$$\frac{\partial W^2}{\partial y}=\frac{\partial W_0}{\partial y}-\frac{1}{2}(y-Y)\,W_1^2+O(\Gamma),$$

Hence on the characteristic $y-Y=x-X$, we have $W^2=W_0$ and

$$\frac{\partial W^2}{\partial x}+\frac{\partial W^2}{\partial y}=\frac{\partial W_0}{\partial x}+\frac{\partial W_0}{\partial y}=(a+b)\,W_0=(a+b)\,W^2;$$

and similarly on $y-Y=-(x-X)$,

$$\frac{\partial W^2}{\partial x}-\frac{\partial W^2}{\partial y}=(a-b)\,W^2.$$

Thus W^2 is the required Riemann-Green function. In theory, we can find the successive coefficients $W_j^{2\alpha+2}$ by a simple recurrence process, and then obtain the Riemann-Green function and the elementary solution at the same time. In practice, this may prove difficult.

For instance, for the equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + \frac{\nu(1-\nu)}{x^2}\, U = 0,$$

it is easy to show that

$$W_0 = 1\,, \qquad W_1^{2\alpha+2} = -\frac{\nu(1-\nu)}{X^2}\, F\left(2, \alpha+1\,;\, \alpha+2\,;\, -\frac{x-X}{X}\right),$$

but I cannot find any simple formula for the higher coefficients for general values of α. In the particular case $\alpha = 0$, we have

$$W_j^2 = (-1)^j \frac{\Gamma(\nu+j)\,\Gamma(1-\nu+j)}{\Gamma(\nu)\,\Gamma(1-\nu)\,(x X)^j}$$

and hence

$$W^2 = F\left(\nu,\ 1-\nu;\ 1;\ -\frac{\Gamma}{4xX}\right) = P_{-\nu}\left(\frac{x^2 - X^2 - (y-Y)^2}{2xX}\right).$$

§ 7. The integral equation method

If we use characteristic variables (r, s), the equation takes the form

$$LU \equiv \frac{\partial^2 U}{\partial r\, \partial s} + A\frac{\partial U}{\partial r} + B\frac{\partial U}{\partial s} + CU = 0$$

where A, B, C are functions of (r, s) alone. The Riemann-Green function $V(r, s; R, S)$ satisfies the adjoint equation

$$L^* V \equiv \frac{\partial^2 V}{\partial r\, \partial s} - \frac{\partial}{\partial r}(AV) - \frac{\partial}{\partial s}(BV) + CV = 0$$

under the conditions

$$\frac{\partial V}{\partial s} = AV \quad \text{on} \quad r = R, \qquad \frac{\partial V}{\partial r} = BV \quad \text{on} \quad s = S$$

and

$$V(R, S; R, S) = 1\,.$$

If V denotes $V(r, s; R, S)$, we then have

$$\int_R^t dr \int_S^u ds \{V_{rs} - (AV)_r - (BV)_s + CV\} = 0\,.$$

Hence

$$V(t,u;R,S) - V(t,S;R,S) - V(R,u;R,S) + V(R,S;R,S) - \int_S^u A(t,s)\, V(t,s;R,S)\, ds +$$

$$+ \int_S^u A(R,s)\, V(R,s;R,S)\, ds - \int_R^t B(r,u)\, V(r,u;R,S)\, dr +$$

$$+ \int_R^t B(r,S)\, V(r,S;R,S)\, dr + \int_R^t dr \int_S^u ds\, C(r,s)\, V(r,s;R,S) = 0\,.$$

Now

$$V(R, S; R, S) = 1\,.$$

Hence, by the condition on the characteristic $s = S$,

$$\int_R^t B(r, S)\, V(r, S; R, S)\, dr = \int_R^t V_r(r, S; R, S)\, dr = V(t, S; R, S) - 1,$$

and similarly

$$\int_S^u A(R, S)\, V(R, s; R, S)\, ds = V(R, u; R, S) - 1.$$

It follows that

$$V(t, u; R, S) = 1 + \int_R^t B(r, u)\, V(r, u; R, S)\, dr +$$
$$+ \int_S^u A(t, s)\, V(t, s; R, S)\, ds - \int_R^t dr \int_S^u ds\, C(r, s)\, V(r, s; R, S).$$

Conversely, it can be shown that, under certain continuity conditions, this integral equation has a unique solution, which is the required Riemann-Green function.

Unfortunately this is not a very good way of actually finding V. Let us consider two examples.

For the equation

$$\frac{\partial^2 U}{\partial r\, \partial s} - \lambda U - 0$$

where λ is a constant, the integral equation reduces to

$$V(r, s; R, S) = 1 + \lambda \int_R^r dt \int_S^s du\, V(t, u; R, S)$$

with a trivial change of notation. If we try to solve this by successive substitutions or (what comes to the same thing) by putting

$$V(r, s; R, S) = \sum_0^\infty \lambda^n f_n(r, s),$$

we get

$$f_0(r, s) = 1$$

$$f_n(r, s) = \int_R^r dt \int_S^s du\, f_{n-1}(t, u).$$

It follows by induction that

$$f_n(r, s) = \frac{(r - R)^n (s - S)^n}{n!\, n!}$$

and hence that

$$V(r, s; R, S) = I_0\{2 \sqrt{[\lambda (r - R)(s - S)]}\}.$$

But if we apply method to

$$\frac{\partial^2 U}{\partial r\, \partial s} + \frac{\alpha(1-\alpha)}{(r+s)^2}\, U = 0$$

where α is a constant, we have

$$V(r, s; R, S) = 1 - \alpha(1-\alpha) \int_R^r dt \int_S^s du\, \frac{V(t, u; R, S)}{(t+u)^2}.$$

The known result is

$$V(r, s; R, S) = P_{-\alpha}\left\{1 + 2\,\frac{(r-R)(s-S)}{(r+s)(R+S)}\right\};$$

but solution by successive substitution gives

$$V = \sum_{0}^{\infty} \alpha^n (\alpha - 1)^n f_n(r, s)$$

where

$$f_0(r, s) = 1$$

$$f_n(r, s) = \int_R^r dt \int_S^s du\, \frac{f_{n-1}(t, u)}{(t+u)^2}.$$

In particular,

$$f_1(r, s) = \log \frac{(R+s)(r+S)}{(r+s)(R+S)}.$$

This is, in fact, the correct first term in a rather unorthodox expansion of the Legendre function. It does not seem possible to get any further in this way. Henrici (*loc. cit.*) attributes the integral equation of this section to I. N. Vekua.

§ 8. Chaundy's papers

In a series of papers written in the five years before the war, Chaundy* discussed in great detail what he called partial differential equations of generalised hypergeometric type, namely equations for the form

$$\prod_1^n f_r(\delta_r)\, Z = x_1 x_2 \ldots x_n \prod_1^n g_r(\delta_r)\, Z$$

where f_r, g_r are polynomials and δ_r denotes the operator $x_r \partial/\partial x_r$; in particular, he found the Riemann-Green function for equations of this type when $n=2$ and the order of the equation is 2. It would take too long to give a complete account of this work. Suffice to say that it involved a very skilful use of symbolic operators and great insight. It should be noted that in Chaundy's work the term involving the highest derivative may be

$$(1 - r s)\,\frac{\partial^2 u}{\partial r\, \partial s}$$

in our notation; this necessitates a slightly more general form of the adjugate equation and an alteration in the definition of the value of the Riemann-Green function at the vertex. It is, however, quite easy to transform his results into our notation. This difficulty does not arise in the example we now give.

In the 1938 paper, Chaundy considered the self-adjoint equation**

$$\frac{\partial^2 u}{\partial r\, \partial s} = \left\{\frac{m_1(m_1+1)}{(r+s)^2} - \frac{m_2(m_2+1)}{(r-s)^2} + \frac{m_3(m_3+1)}{(1-rs)^2} - \frac{m_4(m_4+1)}{(1+rs)^2}\right\} u$$

* Chaundy, T. W.: Quarterly Journ. of Math., Oxford Ser. **6**, 288—303 (1935); **7**, 306—315 (1936); **8**, 280—302 (1937); **9**, 234—240 (1938); **10**, 219—240 (1939); **11**, 101—110 (1940).

** There is a misprint in the enunciation at the beginning of this paper. The sign of the term involving m_3 is wrong.

where m_1, m_2, m_3, m_4 are constants. The Riemann-Green function satisfies this equation and is equal to unity on each of the characteristics $r=R$, $s=S$. He now makes the change of variables

$$x_1 = -\frac{(r-R)(s-S)}{(r+s)(R+S)}, \qquad x_2 = \frac{(r-R)(s-S)}{(r-s)(R-S)},$$

$$x_3 = -\frac{(r-R)(s-S)}{(1-rs)(1-RS)}, \qquad x_4 = \frac{(r-R)(s-S)}{(1+rs)(1+RS)},$$

and shows that the Riemann-Green function satisfies the simultaneous equations

$$\delta_r(\delta_1+\delta_2+\delta_3+\delta_4)\,v = x_r(\delta_r-m_r)(\delta_r+m_r+1)\,v \qquad (r=1,2,3,4)$$

and is equal to unity when $x_1=x_2=x_3=x_4=0$. Such a solution is

$$v = \sum_{r_1,r_2,r_3,r_4=0}^{\infty} \left\{ \prod_{s=1}^{4} (-m_s)_{r_s}(m_s+1)_{r_s} \frac{x_s^{r_s}}{(r_s)!} \right\} \frac{1}{(r_1+r_2+r_3+r_4)!}$$

where $(\alpha)_r=\alpha(\alpha+1)\ldots(\alpha+r-1)$. This he denotes by

$$v = 八\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

If any m_r is zero, the corresponding terms in the multiple power series are replaced by unity.

In particular, if $m_1=m$, $m_2=m_3=m_4=0$, the Riemann-Green function for

$$\frac{\partial^2 u}{\partial r\,\partial s} = \frac{m(m+1)}{(r+s)^2}\,u$$

is

$$v = 八\binom{m}{x_1} = \sum_{r=0}^{\infty} \frac{(-m)_r(m+1)_r}{r!\,r!}\,x_1^r = F(-m, m+1; 1; x_1) = P_m(1-2x_1)$$

or

$$v = P_m\left(1 + 2\frac{(r-R)(s-S)}{(r+s)(R+S)}\right)$$

which is equivalent to (5.9) above.

Again, if $m_1=m$, $m_2=n$, $m_3=m_4=0$, the Riemann-Green function for

$$\frac{\partial^2 u}{\partial r\,\partial s} = \left\{\frac{m(m+1)}{(r+s)^2} - \frac{n(n+1)}{(r-s)^2}\right\} u$$

is

$$v = 八\begin{pmatrix} m & n \\ x_1 & x_2 \end{pmatrix}$$

and this Chaundy recognises to be

$$v = P_m(1-2x_1) - 2x_2\int_0^1 P_m(1-2x_1+2x_1 t)\,P_n'(1-2x_2 t)\,dt$$

which is, in fact, equivalent to (5.13).

§ 9. The use of contour integrals

In his 1936 paper already cited, CHAUNDY obtained an expression for the Riemann-Green function of the equation

$$(r+s)\frac{\partial^2 u}{\partial r\,\partial s} - n\frac{\partial u}{\partial r} - m\frac{\partial u}{\partial s} = 0$$

where m and n are positive integers; he used his method of factorising differential operators and found the Riemann-Green function as a repeated derivative of a rational function. He then observed that this derivative of a rational function could be expressed as a contour integral which had a meaning even when m and n were not positive integers.

Twenty years later, A. G. MACKIE, quite unaware of CHAUNDY'S work, obtained contour integral formulae for the Riemann-Green function of

$$\frac{\partial^2 u}{\partial r \partial s} + \frac{1}{r+s}\left(\alpha\frac{\partial u}{\partial r} + \beta\frac{\partial u}{\partial s}\right) = 0 \quad (9.1)$$

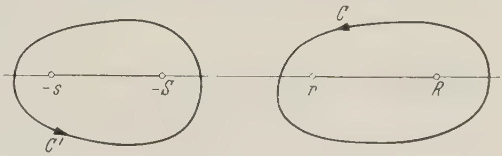

Fig. 8. Contour for Eq. (9.2)

where α, β are constants. He was led to this when extending his work* on the solution of certain gas dynamical problems by complex integrals: so far, MACKIE has not published this result.

The Riemann-Green function associated with the point (R, S) satisfies the adjoint equation *qua* function of (r, s) and the original equation *qua* function of (R, S). It is easily verified that, if z is any complex parameter,

$$(z-R)^{-\beta}(z+S)^{-\alpha}$$

is a particular solution of the original equation and that

$$(r+s)(z-r)^{\beta-1}(z+s)^{\alpha-1}$$

is a particular solution of the adjoint equation. We are thus led to consider a function

$$v(r, s; R, S) = \frac{r+s}{2\pi i}\int_C \frac{(z-r)^{\beta-1}(z+s)^{\alpha-1}}{(z-R)^{\beta}(z+S)^{\alpha}} f(z)\,dz. \qquad (9.2)$$

The integrand has branch points at r, R, $-s$, $-S$; and we can make the integrand one-valued by cutting the z plane along the real axis from $-s$ to $-S$ and from $+r$ to $+R$. We choose the branch that is real for big enough real values of z.

If v is the Riemann-Green function, it must satisfy the conditions

$$v = (R+s)^{\alpha}(R+S)^{-\alpha} \quad \text{on } r = R$$

$$v = (r+S)^{\beta}(R+S)^{-\beta} \quad \text{on } s = S.$$

We take C to be the contour shown in Fig. 8, and see whether we can choose $f(z)$ so as to satisfy these conditions.

* MACKIE, A. G.: Proc. Camb. Phil. Soc. **50**, 131–138 (1954). — Journ. Rat. Mech. Anal. **4**, 733–750 (1955). — Proc. Roy. Soc. A **236**, 265–277 (1956).

Now when $r = R$, we have

$$v(R, s; R, S) = \frac{R+s}{2\pi i} \int_{C_0} \frac{(z+s)^{\alpha-1}}{(z+S)^{\alpha}} f(z) \frac{dz}{z-R}$$

where C_0 is now a simple closed contour surrounding R; the singularities $-s$, $-S$ lie outside C_0. If $f(z)$ is regular inside and on C_0, we have

$$v(R, s; R, S) = \frac{(R+s)^{\alpha}}{(R+S)^{\alpha}} f(R)$$

which is what we need if $f(R)$ is unity. Thus

$$v(r, s; R, S) = \frac{r+s}{2\pi i} \int_{C} \frac{(z-r)^{\beta-1}(z+s)^{\alpha-1}}{(z-R)^{\beta}(z+S)^{\alpha}} dz \tag{9.3}$$

satisfies the conditions on $r = R$.

To deal with the condition on $s = S$, we observe that

$$v(r, s; R, S) = -\frac{r+s}{2\pi i} \int_{C'} \frac{(z-r)^{\beta-1}(z+s)^{\alpha-1}}{(z-R)^{\beta}(z+S)^{\alpha}} dz$$

where C' is as shown in Fig. 8; this follows from the fact that

$$\int_{|z|=\varrho} \frac{(z-r)^{\beta-1}(z+s)^{\alpha-1}}{(z-R)^{\beta}(z+S)^{\alpha}} dz \to 0$$

as $\varrho \to \infty$. If we now put $s = S$, we find that the second characteristic condition is also satisfied. Hence the Riemann-Green function is, in fact, given by equation (9.3).

If we make the substitution

$$z - r = \frac{(R-r)(r+S)t}{(R+S) - (R-r)t},$$

equation (9.3) becomes

$$v(r, s; R, S) = \frac{(r+s)^{\alpha}(r+S)^{\beta-\alpha}}{(R+S)^{\beta}} \frac{1}{2\pi i} \int_{\Gamma} t^{\beta-1}(t-1)^{-\beta}(1+Xt)^{\alpha-1} dt$$

where

$$X = \frac{(R-r)(S-s)}{(R+S)(r+s)}$$

and Γ is a simple closed contour containing the points 0 and 1 but not $-1/X$. From this follows that*

$$v(r, s; R, S) = \frac{(r+s)^{\alpha}(r+S)^{\beta-\alpha}}{(R+S)^{\beta}} F\left(1-\alpha, \beta; 1; -\frac{(R-r)(S-s)}{(R+S)(r+s)}\right). \tag{9.4}$$

As Chaundy pointed out, the method is capable of generalisation, and in his 1939 paper he obtained a contour integral representation of the Riemann-Green function for

$$\frac{\partial^2 U}{\partial x^2} + \frac{2\mu+1}{x} \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} + \frac{2\nu+1}{y} \frac{\partial U}{\partial y}$$

* This can be expressed in a symmetrical form by using the formula connecting $F(a, b; c; x)$ with the hypergeometric functions of argument $1-x$.

where μ, ν are constants, *viz.*

$$\frac{2i}{\pi^3}\left(\frac{x}{X}\right)^{\mu}\left(\frac{y}{Y}\right)^{\nu}\int^{(0+)} K_\mu(-x\,t)\,K_\nu(-y\,t)\,K_\mu(X\,t)\,K_\nu(Y\,t)\,t\,dt.$$

In the first instance, $\mu-\frac{1}{2}$ and $\nu-\frac{1}{2}$ are integers; but the result is extended to any μ and ν.

§ 10. Titchmarsh's derivation of $v\,(r, s; R, S)$ for the equation of damped waves*

If we use the characteristic variables r, s, the problem is to find the solution of

$$\frac{\partial^2 v}{\partial r\,\partial s}=v$$

which has the value unity on the characteristics $r=R$ and $s=S$. If we put $r-R=x$, $s-S=y$, we have to find the solution of

$$\frac{\partial^2 v}{\partial x\,\partial y}=v$$

which is equal to unity on the axes of coordinates.

Let us write

$$V(\zeta, y)=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty v\,(x, y)\,e^{i\zeta x}\,dx$$

where $\zeta=\xi+i\eta$ and $\eta>c>0$. Then

$$\frac{\partial V}{\partial y}=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty \frac{\partial v}{\partial y}\,e^{i\zeta x}\,dx=\frac{1}{\sqrt{(2\pi)}}\left[\frac{\partial v}{\partial y}\,e^{i\zeta x}\right]_0^\infty-\frac{1}{i\zeta\sqrt{(2\pi)}}\int_0^\infty \frac{\partial^2 v}{\partial x\,\partial y}\,e^{i\zeta x}\,dx.$$

But $\partial v/\partial y$ vanishes when $x=0$. Hence

$$\frac{\partial V}{\partial y}=-\frac{1}{i\zeta\sqrt{(2\pi)}}\int_0^\infty \frac{\partial^2 v}{\partial x\,\partial y}\,e^{i\zeta x}\,dx=-\frac{1}{i\zeta\sqrt{(2\pi)}}\int_0^\infty v\,e^{i\zeta x}\,dx$$

so that

$$\frac{\partial V}{\partial y}=-\frac{V}{i\zeta}.$$

This gives

$$V(\zeta, y)=A\,(\zeta)\,e^{iy/\zeta}.$$

Now make y tend to zero. Then

$$A\,(\zeta)=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty v\,(x, 0)\,e^{i\zeta x}\,dx=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty e^{i\zeta x}\,dx=-\frac{1}{i\zeta\sqrt{(2\pi)}}.$$

It follows that

$$v\,(x, y)=-\frac{1}{2\pi i}\int_{ik-\infty}^{ik+\infty} e^{iy/\zeta}\,e^{-ix\zeta}\,\frac{d\zeta}{\zeta}=I_0\big(2\sqrt{(x\,y)}\big)$$

and hence that

$$v\,(r, s; R, S)=I_0\big(2\sqrt{\{(r-R)\,(s-S)\}}\big).$$

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* Titchmarsh, E. C.: Theory of Fourier Integrals, pp. 297—298. Oxford 1937.

Although the equation of damped waves is very special, this work of Titchmarsh's is of interest in that it is a direct solution of the characteristic boundary value problem satisfied by the Riemann-Green function, and is the only one of which I am aware.

§ 11. Conclusion

In this report I have, I believe, covered all the known methods of finding the Riemann-Green function and have listed all the known cases, apart from trivial changes of the dependent or independent variables. I should like to take this opportunity of thanking Harvard University and the staff of the Harvard Department of Mathematics for their kind hospitality during the Fall Semester of 1957—58, which gave me the leisure to complete this work, and also Dr. A. G. Mackie of the University of St. Andrews for the many interesting discussions we had concerning the contour integral method of § 9.

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On a Singular Boundary Value Problem for an Equation of Hyperbolic Type

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§ 1. Introduction

One of the ways of solving the initial value problem for the equation of wave motions involves the use of spherical means. For instance, if $u(x, y, z, t)$ satisfies $\nabla^2 u = \partial^2 u/\partial t^2$ under the initial conditions $u = f(x, y, z)$, $\partial u/\partial t = 0$, the spherical mean $\bar{u}(r, t)$ of u over a sphere of radius r and centre at an arbitrary fixed point (x, y, z) satisfies the equation

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{u}}{\partial r} = \frac{\partial^2 \bar{u}}{\partial t^2}$$

under the initial conditions $\bar{u}(r, 0) = \bar{f}(r)$, $\bar{u}_t(r, 0) = 0$, in an obvious notation. It readily follows that

$$\bar{u}(r, t) = \frac{(t+r)\bar{f}(t+r) - (t-r)\bar{f}(t-r)}{2r}$$

and hence that

$$u(x, y, z, t) = \lim_{r\to 0} \bar{u}(r, t) = \frac{\partial}{\partial t}\{t\bar{f}(t)\}$$

the well-known Poisson solution. The problem of solving the wave-equation is thus reduced to that of finding $\lim\limits_{r\to 0} \bar{u}(r, t)$ given $\bar{u}(r, 0)$.

The same method can be applied to the special case of the Euler-Poisson-Darboux equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2} + \frac{2}{t}\frac{\partial u}{\partial t}$$

under the same initial conditions. In this case

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r}\frac{\partial \bar{u}}{\partial r} = \frac{\partial^2 \bar{u}}{\partial t^2} + \frac{2}{t}\frac{\partial \bar{u}}{\partial t}$$

under the conditions $\bar{u} = \bar{f}(r)$, $\bar{u}_t = 0$ when $t = 0$. The problem here has a symmetry, since $\bar{u}_r \to 0$ as $r \to 0$. This suggests that

$$\bar{u}(r, t) \to \bar{u}(t, 0)$$

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as $r\to 0$, and hence that $$u(x, y, z, t) = \bar{f}(t),$$

which is, in fact, ASGEIRSSON'S solution.

If one wished to extend this to the general Euler-Poisson-Darboux equation, *viz.*

$$\sum_{1}^{m} \frac{\partial^2 u}{\partial x_k^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\lambda}{t} \frac{\partial u}{\partial t}, \tag{1.1}$$

one would to consider the equation

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{m-1}{r} \frac{\partial \bar{u}}{\partial r} = \frac{\partial^2 \bar{u}}{\partial t^2} + \frac{\lambda}{t} \frac{\partial \bar{u}}{\partial t};$$

and it is this equation which we consider here for general values of the constants λ and m. It is convenient to change the notation slightly, and to discuss the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{2\alpha}{x} \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y} \frac{\partial U}{\partial y}$$

where α and β are positive constants. Actually we shall assume that α and β are large enough to ensure the absolute convergence of all the integrals which occur. We can then deal with smaller values of α and β by analytical continuation, just as DIAZ & WEINBERGER* did in the case of the Euler-Poisson-Darboux equation.

This partial differential equation looks rather special. The point about it is that it possesses two singular lines. Apart from the extensive work on the Euler-Poisson-Darboux equation, very little is known about partial differential equations with singular coefficients; a discussion of this special problem may serve to suggest the direction in which further progress may be made.

§ 2. The problem

The problem is

To find a solution of

$$\frac{\partial^2 U}{\partial x^2} + \frac{2\alpha}{x} \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2} + \frac{2\beta}{y} \frac{\partial U}{\partial y} \tag{2.1}$$

where α, β are positive constants, such that

(i) *U and its first derivatives are continuous in $x \geqq 0$, $y \geqq 0$,*

(ii) *the second derivatives of U are continuous in $x > 0$, $y > 0$,*

(iii) *$U = f(x)$ when $y = 0$, $x \geqq 0$; $U = g(y)$ when $x = 0$, $y \geqq 0$; $f(0) = g(0)$.*

At first sight it looks as though there are insufficient data, since in a mixed problem (such as the vibrations of a semi-infinite string with a fixed end) one can assign two data on $y = 0$, $x \geqq 0$ and one datum on $x = 0$, $y \geqq 0$. The fact that the data are given on singular lines of the differential equation upsets all this. By a simple extension of the arguments recently used by W. WALTER**, it can be shown that the problem has no solution unless $U_y = 0$ on $y = 0$, $x \geqq 0$ and $U_x = 0$ on $x = 0$, $y \geqq 0$. Thus we have too much data, not too little. Moreover, if this problem has a solution, the energy-integral type of argument shows that it is unique.

* DIAZ & WEINBERGER: Proc. Amer. Math. Soc. **4**, 703—715 (1953).

** WALTER, W.: Math. Z. **67**, 361—376 (1957).

As things stand, the problem has no solution at all unless $f(x)$ and $g(y)$ are suitably related. This can be seen at once by considering domains of influence. For the data $U=f(x)$, $U_y=0$ on $y=0$, $x>0$, determine a solution in $0<y\leqq x$; the data $U=g(y)$, $U_x=0$ on $x=0$, $y>0$, determine a solution in $0<x\leqq y$. Continuity across the characteristic $y=x$ leads to an integral equation connecting $f(x)$ and $g(y)$, and hence $g(y)$ can be expressed in terms of $f(x)$. This, of course, is precisely what one needs if one tries to solve equation (1.1) by spherical means.

§ 3. Polynomial boundary data

If $f(x)$ and $g(y)$ are suitably related polynomials, it does not follow that there is an analytic solution, still less a polynomial solution. Take the simplest case, when $f(x)$ is a positive integral power of x, say

$$f(x)=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2}+\frac{1}{2}N)}\,x^N, \tag{3.1}$$

the coefficient being introduced to simplify matters later. The corresponding solution must be a homogeneous function of degree N, and it is easily shown to be*

$$U=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2}+\frac{1}{2}N)}\,x^N F\Big(-\frac{1}{2}N,\frac{1}{2}-\frac{1}{2}N-\alpha;\beta+\frac{1}{2};\frac{y^2}{x^2}\Big) \tag{3.2}$$

which, incidentally, automatically satisfies the condition $U_y=0$ when $y=0$.

There are two cases. If N is an even integer, (3.2) is a polynomial of degree N, so that this solution holds in the whole quadrant $x\geqq 0$, $y\geqq 0$; and the value taken on $x=0$, $y\geqq 0$ is

$$g(y)=\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+\frac{1}{2}+\frac{1}{2}N)}\,y^N. \tag{3.3}$$

And we observe that, if $\beta>\alpha$,

$$y^{2\beta-1}g(y)=\frac{2\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha)}\int\limits_0^y x^{2\alpha-1}(y^2-x^2)^{\beta-\alpha-1}f(x)\,x\,dx \tag{3.4}$$

or

$$t^{\beta-\frac{1}{2}}g(\sqrt{t})=\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}\,I^{\beta-\alpha}\,t^{\alpha-\frac{1}{2}}f(\sqrt{t}) \tag{3.5}$$

where $I^{\beta-\alpha}$ denotes the Riemann-Liouville operator of integration of fractional order.

But if N is odd, the hypergeometric series in (3.2) converges only if $y^2/x^2\leqq 1$, and so this solution is valid only in half the quadrant. It might be thought that we could continue the solution (3.2) analytically, regarding it as a function of the complex variable $z=y^2/x^2$. But the hypergeometric function has a branch point at $z=1$, and the analytical continuation (which is in general not real) depends on how one avoids this branch point—it is not a unique continuation. If, however, we start again in the case when N is odd, with $g(y)$ given by (3.3), we get another solution:

$$U=\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta+\frac{1}{2}+\frac{1}{2}N)}\,y^N F\Big(-\frac{1}{2}N,\frac{1}{2}-\frac{1}{2}N-\beta;\alpha+\frac{1}{2};\frac{x^2}{y^2}\Big) \tag{3.6}$$

* The other solution of the hypergeometric equation, that of the type valid in $|z|<1$, $z^{1-c}F(1+a-c,1+b-c;2-c;z)$, does not occur since it would make U_y infinite when $y=0$.

valid in $x^2/y^2 \leqq 1$. Although (3.6) is not the analytical continuation of (3.2), (3.2) and (3.6) together define a function which is continuous in the first quadrant; and, moreover, the first and second derivatives are continuous across $y = x$ for any integer N provided that $\alpha + \beta > 1$. Thus with polynomial or analytic data connected by (3.4) we can build up solutions with continuous second derivatives by using the polynomial solutions of even degree and the hypergeometric solutions of odd degree.

§ 4. Continuously differentiable data

If $f(x)$ and $g(y)$ are merely continuously differentiable functions, the previous method fails, and we have recourse to RIEMANN's method. This gives* us

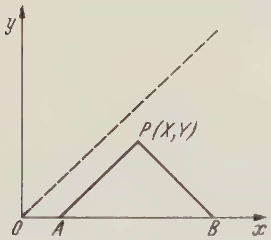

Fig. 1. Diagram for RIEMANN's method

$$U(X,Y) = \frac{1}{2}[UV]_A + \frac{1}{2}[UV]_B + \\ + \frac{1}{2}\int_{AB}\left\{VU_y - UV_y + \frac{2\beta}{y}UV\right\}dx + \\ + \left\{VU_x - UV_x + \frac{2\alpha}{x}UV\right\}dy \tag{4.1}$$

where $V(x, y; X, Y)$ is the Riemann-Green function and A and B are the points where the curve carrying the data is cut by the characteristics $y - x = Y - X$ and $y + x = Y + X$ respectively.

Now if $Y > y > 0$ and $X - x$ lies between $\pm(Y - y)$, it is known** that, if $\beta - \frac{1}{2}$ is not an integer,

$$V(x, y; X, Y) = \frac{\pi}{\cos\pi\beta} \cdot \frac{x^{\alpha+\frac{1}{2}}\, y^{\beta+\frac{1}{2}}}{X^{\alpha-\frac{1}{2}}\, Y^{\beta-\frac{1}{2}}} \times \\ \times \int_0^\infty \lambda J_{\alpha-\frac{1}{2}}(\lambda x) J_{\alpha-\frac{1}{2}}(\lambda X)\left\{J_{\beta-\frac{1}{2}}(\lambda y) J_{\frac{1}{2}-\beta}(\lambda Y) - J_{\beta-\frac{1}{2}}(\lambda Y) J_{\frac{1}{2}-\beta}(\lambda y)\right\} d\lambda. \tag{4.2}$$

If we take the case $0 < Y \leqq X$, the situation is as in Fig. 1, A being $(X - Y, 0)$, B being $(X + Y, 0)$. But as the datum $f(x)$ is given on the singular line $y = 0$, we have to consider the limiting forms of V and V_y as $y \to 0$. Since $\beta > 0$, V tends to zero as $y \to 0$. *If we assume that* $\beta > \frac{1}{2}$, we find that, for *all* such β,

$$\frac{V}{y} \to \frac{\Gamma(\beta - \frac{1}{2})}{2^{\frac{1}{2}-\beta}} \frac{x^{\alpha+\frac{1}{2}}}{X^{\alpha-\frac{1}{2}}\, Y^{\beta-\frac{1}{2}}} \int_0^\infty \lambda^{\frac{3}{2}-\beta} J_{\alpha-\frac{1}{2}}(\lambda x) J_{\alpha-\frac{1}{2}}(\lambda X) J_{\beta-\frac{1}{2}}(\lambda Y)\, d\lambda \\ = \frac{\Gamma(\beta - \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\beta)} \frac{x^\alpha R_0^{2\beta-2}}{X^\alpha Y^{2\beta-1}} F\left(\alpha, 1-\alpha; \beta; \frac{R_0^2}{4xX}\right) \tag{4.3}$$

where

$$R_0^2 = Y^2 - (X - x)^2$$

by a formula due to H. M. MACDONALD***. This holds when x lies between $X \pm Y$ if $0 < Y \leqq X$; but if $0 < X \leqq Y$, it holds only when x lies between $Y \pm X$. Since V vanishes on $y = 0$, (4.3) also gives the value of V_y when $y = 0$.

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* See, for example, eqn. (2.3) of the preceding paper by E. T. COPSON.
** *ibid.*, eqn. (5.12).
*** See WATSON, G. N.: Theory of Bessel Functions, p. 411—412. Cambridge 1922.

It follows immediately that, if $2\beta > 1$ and $0 < Y \leqq X$,

$$U(X, Y) = \frac{\Gamma(\beta + \frac{1}{2})}{\pi^{\frac{1}{2}}\, \Gamma(\beta)\, X^{\alpha}\, Y^{2\beta-1}} \int\limits_{X-Y}^{X+Y} f(x)\, x^{\alpha}\, R_0^{2\beta-2}\, F\left(\alpha, 1-\alpha; \beta; \frac{R_0^2}{4xX}\right) dx. \tag{4.4}$$

Similarly, if $2\alpha > 1$ and $0 < X \leqq Y$

$$U(X, Y) = \frac{\Gamma(\alpha + \frac{1}{2})}{\pi^{\frac{1}{2}}\, \Gamma(\alpha)\, X^{2\alpha-1}\, Y^{\beta}} \int\limits_{Y-X}^{Y+X} g(y)\, y^{\beta}\, R_1^{2\alpha-2}\, F\left(\beta, 1-\beta; \alpha; \frac{R_1^2}{4yY}\right) dy \tag{4.5}$$

where

$$R_1^2 = X^2 - (Y-y)^2.$$

We thus have one solution depending only on $f(x)$ valid in $0 < Y \leqq X$, a second depending only on $g(y)$ valid in $0 < X \leqq Y$. We have now to see whether we can fit them together in such a way as to form one solution with continuous first and second derivatives valid in the whole quadrant. Actually, as we shall see later, it suffices to make U continuous.

Now from (4.4) we have

$$\begin{aligned} U(X, X) &= \frac{\Gamma(\beta + \frac{1}{2})\, 2^{\alpha+2\beta-1}}{\pi^{\frac{1}{2}}\, \Gamma(\beta)} \int\limits_0^1 f(2Xt)\, t^{\alpha+\beta-1} (1-t)^{\beta-1}\, F\left(\alpha, 1-\alpha; \beta; \frac{1-t}{2}\right) dt \\ &= \frac{\Gamma(\beta + \frac{1}{2})\, 2^{\alpha+2\beta-1}}{\sqrt{\pi}} \int\limits_0^1 f(2Xt)\, t^{\alpha+\beta-1} (1-t^2)^{\frac{1}{2}\beta-\frac{1}{2}}\, P_{-\alpha}^{1-\beta}(t)\, dt \end{aligned}$$

and from (4.5)

$$U(X, X) = \frac{\Gamma(\alpha + \frac{1}{2})\, 2^{2\alpha+\beta-1}}{\sqrt{\pi}} \int\limits_0^1 g(2Xt)\, t^{\alpha+\beta-1} (1-t^2)^{\frac{1}{2}\alpha-\frac{1}{2}}\, P_{-\beta}^{1-\alpha}(t)\, dt.$$

Thus f and g are connected by the equation*

$$\begin{aligned} 2^{\beta}\, \Gamma(\beta + \tfrac{1}{2}) \int_0^1 f(Xt)\, t^{\alpha+\beta-1} (1-t^2)^{\frac{1}{2}\beta-\frac{1}{2}}\, P_{-\alpha}^{1-\beta}(t)\, dt \\ = 2^{\alpha}\, \Gamma(\alpha + \tfrac{1}{2}) \int_0^1 g(Xt)\, t^{\alpha+\beta-1} (1-t^2)^{\frac{1}{2}\alpha-\frac{1}{2}}\, P_{-\beta}^{1-\alpha}(t)\, dt. \end{aligned} \tag{4.6}$$

If $\alpha = \beta$, $f(x)$ and $g(x)$ are identical. When $\beta > \alpha$, we should expect that

$$y^{2\beta-1}\, g(y) = \frac{2\Gamma(\beta + \frac{1}{2})}{\Gamma(\alpha + \frac{1}{2})\, \Gamma(\beta - \alpha)} \int\limits_0^y x^{2\alpha-1} f(x)\, (y^2 - x^2)^{\beta-\alpha-1}\, x\, dx \tag{4.7}$$

and this is, in fact, the case. It can be proved by substituting for $g(Xt)$ from (4.7) in the right-hand side of (4.6), inverting the order of integration and using the result

$$\begin{aligned} \int_u^1 t^{\alpha-\beta} (1-t^2)^{\frac{1}{2}\alpha-\frac{1}{2}} (t^2-u^2)^{\beta-\alpha-1}\, P_{-\beta}^{1-\alpha}(t)\, dt \\ = 2^{\beta-\alpha-1}\, \Gamma(\beta-\alpha)\, u^{\beta-\alpha-1} (1-u^2)^{\frac{1}{2}\beta-\frac{1}{2}}\, P_{-\alpha}^{1-\beta}(u). \end{aligned}$$

* We have replaced $2X$ by X.

I can give no reference for this last identity. It can be proved by showing that, for all positive values of s,

$$\int_0^1 u^{2\alpha+s} \int_u^1 t^{\alpha-\beta}(1-t^2)^{\frac{1}{2}\alpha-\frac{1}{2}}(t^2-u^2)^{\beta-\alpha-1} P_{-\beta}^{1-\alpha}(t)\,dt$$
$$= 2^{\beta-\alpha-1}\Gamma(\beta-\alpha)\int_0^1 u^{\alpha+\beta+s-1}(1-u^2)^{\frac{1}{2}\beta-\frac{1}{2}} P_{-\alpha}^{1-\beta}(u)\,du$$

and then using Lerch's Theorem. We show in the next section that the matching of the two solutions along $y=x$ does suffice to imply the matching of the first and second derivatives.

Since (4.7) can be written in the form

$$g(y) = \frac{2\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha)} \int_0^1 t^{2\alpha}(1-t^2)^{\beta-\alpha-1} f(y\,t)\,dt$$

the condition $f(0)=g(0)$ of the problem is satisfied. Moreover*, if $\beta-\alpha \geqq 1$, this relation and the fact that $f(x)$ is continuously differentiable imply that $g(y)$ has continuous derivatives of order higher than unity, the actual order depending on the difference $\beta-\alpha$. Conversely, if we have merely $\beta-\alpha>0$, the continuous differentiability of $g(y)$ does not imply the existence of a continuously differentiable $f(x)$ satisfying (4.7). Thus if $\beta>\alpha$, the continuously differentiable datum must be assigned on the axis of x.

§ 5. The continuity of the derivatives

In §4 we found two solutions, U_1 and U_2 say, of equation (2.1) valid respectively in $0<y\leqq x$ and in $0<x\leqq y$, such that U_1 and U_2 were identical on $y=x$. We now show that if $\alpha+\beta$ is big enough, the first and second derivatives of U_1 and U_2 are the same on $y=x$.

If we introduce the characteristic variables $r=x-y$, $s=x+y$, we have two solutions of

$$U_{rs} + \left(\frac{\alpha}{s+r} + \frac{\beta}{s-r}\right) U_r + \left(\frac{\alpha}{s+r} - \frac{\beta}{s-r}\right) U_s = 0$$

valid respectively on $s>r\geqq 0$ and on $s>-r\geqq 0$ and identical on $r=0$. The last condition implies that $\partial U_1/\partial s$ and $\partial U_2/\partial s$ are equal on $r=0$.

Denote the values of $\partial U/\partial r$ and $\partial U/\partial s$ on $r=0$ by $u(s)$ and $v(s)$; then

$$\frac{du}{ds} + \frac{\alpha+\beta}{s} u = -\frac{\alpha-\beta}{s} v.$$

Hence, if c is any positive constant,

$$u(s)\, s^{\alpha+\beta} = u(c)\, c^{\alpha+\beta} - (\alpha-\beta)\int_c^s t^{\alpha+\beta-1} v(t)\,dt.$$

By hypothesis $u(s)$ and $v(s)$ are continuous in any finite interval $0\leqq s\leqq S$, and are therefore bounded. And as $\alpha+\beta>0$, we can make c tend to zero, and obtain

$$u(s)\, s^{\alpha+\beta} = -(\alpha-\beta)\int_0^s t^{\alpha+\beta-1} v(t)\,dt.$$

* I owe these remarks to Professor Erdélyi.

Hence

$$[u(s)]_1^2\, s^{\alpha+\beta} = -(\alpha-\beta)\int_0^s t^{\alpha+\beta-1}\,[v(t)]_1^2\,dt = 0\,.$$

Hence $\partial U_1/\partial r$ and $\partial U_2/\partial r$ have the same values on $r=0$, so that the first derivatives of U_1 and U_2 are identical on $y=x$.

Evidently the second derivatives $\partial^2 U_1/\partial r\,\partial s$ and $\partial^2 U_2/\partial r\,\partial s$ have the same values on $r=0$ by the differential equation; so also have $\partial^2 U_1/\partial s^2$ and $\partial^2 U_2/\partial s^2$. To deal with $\partial^2 U/\partial r^2$, we have to assume the existence of third derivatives. It can be shown that if the value of $\partial^2 U/\partial r^2$ on $r=0$ is denoted by $w(s)$, then

$$\frac{dw}{ds} + \frac{\alpha+\beta}{s}\,w = \frac{\alpha^2-\beta^2+\alpha-\beta}{s^2}\,u + \frac{(\alpha-\beta)^2+\alpha+\beta}{s^2}\,v\,.$$

Hence

$$w(s)\,s^{\alpha+\beta} = w(c)\,c^{\alpha+\beta} + \int_c^s t^{\alpha+\beta-2}\left\{(\alpha^2-\beta^2+\alpha-\beta)\,u(t) + [(\alpha-\beta)^2+\alpha+\beta]\,v(t)\right\}dt\,.$$

If $\alpha+\beta>1$, we can make c tend to zero provided that we assume $w(c)$ bounded as $c\to 0$. The equality of the values of the second derivatives on $r=0$ will then follow provided we pay the price of further assumptions on the second derivatives. Fortunately, as we show in the next section, it is not necessary to do this.

§ 6. An alternative method

An alternative method of dealing with the case $Y>X$ of § 4 is suggested by that used for solving the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

when the data are

$$u = f(x)\,,\qquad \frac{\partial u}{\partial y} = g(x)\qquad y=0,\quad x\geqq 0$$

$$u = F(y)\qquad\qquad x=0,\quad y\geqq 0\,.$$

We suppose that the data imply that

$$\frac{\partial u}{\partial x} = G(y)\qquad x=0,\quad y\geqq 0$$

where $G(y)$ is an unknown function. For such a solution

$$\int_\Gamma u_y\,dx + u_x\,dy = 0$$

for any closed curve Γ. If we take Γ to be $OBPAO$ of Fig. 2, we readily find that

$$u(X,Y) = \tfrac{1}{2}f(Y+X) + \tfrac{1}{2}F(Y-X) + \tfrac{1}{2}\int_0^{Y+X} g(x)\,dx - \tfrac{1}{2}\int_0^{Y-X} G(y)\,dy$$

which involves the unknown function $G(y)$. But if we take Γ to be the triangle OCA, we get

$$0 = \tfrac{1}{2}f(Y-X) - \tfrac{1}{2}F(Y-X) + \tfrac{1}{2}\int_0^{Y-X} g(x)\,dx - \tfrac{1}{2}\int_0^{Y-X} G(y)\,dy\,.$$

And, if we subtract, we find that, when $Y>X>0$,

$$u(X,Y) = \tfrac{1}{2}f(Y+X) - \tfrac{1}{2}f(Y-X) + F(Y-X) + \tfrac{1}{2}\int_{Y-X}^{Y+X} g(x)\,dx\,.$$

For the equation (2.1), everything turns out to be even simpler. We assume that $Y > X > 0$ and apply formula (4.1) to the contour $OBPAO$ of Figure 2. By (4.2), V behaves like a multiple of $x^{2\alpha}$ when x is small; hence *if we assume that* $\alpha > \frac{1}{2}$, the integral along AO and the point A make no contribution, and we are left with

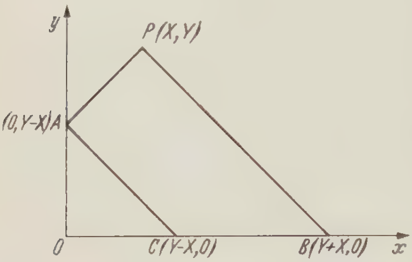

Fig. 2. Contour for the second method

$$u(X,Y) = \frac{1}{2}[UV]_B + \\ + \frac{1}{2}\int_{OB}\left\{VU_y - UV_y + \frac{2\beta}{y}UV\right\}dx.$$

As before, V tends to zero as y tends to zero, and

$$\frac{V}{y} \to \frac{\Gamma(\beta-\frac{1}{2})}{2^{\frac{1}{2}-\beta}}\,\frac{x^{\alpha+\frac{1}{2}}}{X^{\alpha-\frac{1}{2}}Y^{\beta-\frac{1}{2}}}\int_0^\infty \lambda^{\frac{3}{2}} J_{\alpha-\frac{1}{2}}(\lambda x)\,J_{\alpha-\frac{1}{2}}(\lambda X)\,J_{\beta-\frac{1}{2}}(\lambda Y)\,dY$$

if $\beta > \frac{1}{2}$. When $Y - X < x < Y + X$, the value of this integral is given by (4.3); but when $0 < x < Y - X$,

$$\frac{V}{y} \to \frac{2\Gamma(\beta-\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha)}\cdot\frac{x^{2\alpha}(Y^2-X^2-x^2)^{\beta-\alpha-1}}{Y^{2\beta-1}}\times \\ \times F\left(\frac{\alpha-\beta+2}{2}, \frac{\alpha-\beta+1}{2}; \alpha+\frac{1}{2}; \frac{4x^2X^2}{(Y^2-X^2-x^2)^2}\right).$$

It follows then, that, if α and β are greater than $\frac{1}{2}$, the value of $U(X,Y)$ when $Y \geqq X > 0$ is

$$U(X,Y) = \frac{2\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha)\,Y^{2\beta-1}}\int_0^{Y-X} f(x)\,x^{2\alpha}(Y^2-X^2-x^2)^{\beta-\alpha-1}\times \\ \times F\left(\frac{\alpha-\beta+2}{2}, \frac{\alpha-\beta+1}{2}; \alpha+\frac{1}{2}; \frac{4x^2X^2}{(Y^2-X^2-x^2)^2}\right)dx + \\ + \frac{\Gamma(\beta+\frac{1}{2})}{\pi^{\frac{1}{2}}\,\Gamma(\beta)\,X^\alpha\,Y^{2\beta-1}}\int_{Y-X}^{Y+X} f(x)\,x^\alpha R_0^{2\beta-1}F\left(\alpha, 1-\alpha; \beta; \frac{R_0^2}{4xX}\right)dx. \tag{6.1}$$

This is the required solution when $Y \geq X$, and should be compared with (4.4) for $Y \leqq X$.

Lastly, when $\beta > \alpha$, we may make X tend to zero in (6.1). It is readily seen that the second line tends to zero, and we are left with

$$g(Y) = U(0,Y) = \frac{2\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha)\,Y^{2\beta-1}}\int_0^Y f(x)\,x^{2\alpha}(Y^2-x^2)^{\beta-\alpha-1}dx$$

which is, in fact, precisely equation (4.7).

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On an Analogue of the Euler-Cauchy Polygon Method for the Numerical Solution of $u_{xy}=f(x,y,u,u_x,u_y)$

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Abstract. This paper[1] develops, with an eye on the numerical applications, an analogue of the classical Euler-Cauchy polygon method (which is used in the solution of the ordinary differential equation

$$\frac{dy}{dx}=f(x,y), \quad y(x_0)=y_0)$$

for the solution of the following characteristic boundary value problem for a hyperbolic partial differential equation

$$u_{xy}=f(x,y,u,u_x,y_y),$$

$$u(x,y_0)=\sigma(x),$$

$$u(x_0,y)=\tau(y),$$

where $\sigma(x_0)=\tau(y_0)$. The method presented here, which may be roughly described as a process of bilinear interpolation, has the advantage over previously proposed methods that only the tabulated values of the given functions $\sigma(x)$ and $\tau(y)$ are required for its numerical application. Particular attention is devoted to the proof that a certain sequence of approximating functions, constructed in a specified way, actually converges to a solution of the boundary value problem under consideration. Known existence theorems are thus proved by a process which can actually be employed in numerical computation.

[1] This paper was issued on 16 January 1957 as NAVORD Report 4451, U. S. Naval Ordnance Laboratory, White Oak, Maryland, and was presented to the American Mathematical Society in October 1956.

§ 1. Introduction

The classical initial value problem for the ordinary differential equation

$$\frac{dy}{dx} = f(x, y),$$

(where the real valued continuous function $f(x, y)$ is defined for $x_0 \leqq x \leqq x_0 + a$ and $-\infty < y < +\infty$) consists in the determination of a real valued function $y(x)$, defined on $x_0 \leqq x \leqq x_0 + a$, which satisfies the given ordinary differential equation on this interval, and also satisfies the initial condition

$$y(x_0) = y_0,$$

where y_0 is a given real number.

Among the many methods which have been employed for proving the existence of a solution $y(x)$ to this problem, mention will be made here only of PICARD's method of successive approximations (see *e.g.*, G. SANSONE [*21*, vol. I, pp. 9—14], E. L. INCE [*12*, pp. 63—65], E. A. CODDINGTON & N. LEVINSON [*28*, p. 11—13], or E. KAMKE [*16*, pp. 51—56]); of L. TONELLI's method (see, *e.g.*, L. TONELLI [*13*], G. SANSONE [*21*, vol. I, pp. 45—48]); and of the Euler-Cauchy polygon method (see, *e.g.*, G. SANSONE [*21*, vol. I, pp. 36—45, vol. II, pp. 208—283], E. L. INCE [*12*, pp. 75—81], E. A. CODDINGTON & N. LEVINSON [*28*, pp. 3—7], E. KAMKE [*16*, pp. 62—64], or G. A. BLISS [*9*, pp. 86—92]).

For the numerical purpose of the actual construction of a solution the Euler-Cauchy polygon method is usually the most advantageous. The construction of the Euler-Cauchy polygons may be described as follows. For each positive integer m, let

$$x_0 \equiv x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} \equiv x_0 + a,$$

be a subdivision of the interval $x_0 \leqq x \leqq x_0 + a$ into m closed subintervals $x_{k,m} \leqq x \leqq x_{k+1,m}$, where $k = 0, 1, \ldots, m-1$. On each such subinterval the ordinary differential equation is, so to speak, replaced by one whose right-hand side is a (suitably chosen) constant, so that the corresponding function approximating a solution turns out to be a linear function on each subinterval. More precisely put, the polygonal function y_m, which is an approximation to a solution, is defined recurrently by the equations

$$\begin{array}{lll}
\frac{dy_m}{dx}(x) = f(x_{0,m}; y_0), & y_m(x_{0,m}) = y_0, & \text{on } x_{0,m} \leqq x \leqq x_{1,m}, \\
\frac{dy_m}{dx}(x) = f(x_{1,m}; y_1), & y_m(x_{1,m}) = y_1, & \text{on } x_{1,m} \leqq x \leqq x_{2,m}, \\
\vdots & \vdots & \vdots \\
\frac{dy_m}{dx}(x) = f(x_{k,m}; y_k), & y_m(x_{k,m}) = y_k, & \text{on } x_{k,m} \leqq x \leqq x_{k+1,m}, \\
\vdots & \vdots & \vdots
\end{array}$$

for $k = 0, 1, \ldots, m-1$. Notice that, for simplicity in writing these equations, the symbol y_k is used to denote the value of the function $y_m(x)$ at $x_{k,m}$, a value which is obtained from the definition of y_m as a linear function on the preceding subinterval $x_{k-1,m} \leqq x \leqq x_{k,m}$ and which is used as an initial value for the function

$y_m(x)$ for the "miniature" initial value problem (of the same kind as the original one, but whose differential equation has a *constant* right-hand side):

$$\frac{dy_m}{dx}(x) = f(x_{k,m}; y_k), \qquad y_m(x_{k,m}) = y_k,$$

on the next subinterval $x_{k,m} \leqq x \leqq x_{k+1,m}$. For each positive integer m, the function $y_m(x)$ is continuous on the interval $x_0 \leqq x \leqq x_0 + a$, but its derivative will, in general, not exist throughout the interval, since it may jump at the subdivision numbers $x_{k,m}$.

Under the sole additional hypothesis that the function $f(x, y)$ is bounded in absolute value on $x_0 \leqq x \leqq x_0 + a$, $-\infty < y < +\infty$, it follows that the sequence of functions $\{y_m(x)\}$ is equibounded in absolute value and equicontinuous on the interval $x_0 \leqq x \leqq x_0 + a$, and hence, by Ascoli's theorem [*1*] (see also Tonelli [*11*, p. 76—86]) there is a subsequence of the sequence $\{y_m(x)\}$ which converges uniformly to a continuous limit function on $x_0 \leqq x \leqq x_0 + a$. If, further, it is supposed that the maximum length of the subintervals of the subdivision of $x_0 \leqq x \leqq x_0 + a$ approaches zero, *i.e.*

$$\lim_{m \to \infty} \left[\max_{k=0,1,\ldots,m-1} (x_{k+1,m} - x_{k,m}) \right] = 0,$$

then every such continuous limit function is a solution of the original initial value problem, whose solution need not be unique. (It should be noticed that the condition on the maximum length of the subintervals is automatically satisfied in the most common case when the m^{th} subdivision consists of m subintervals of equal length, namely a/m.) If, besides this, the function $f(x, y)$ satisfies a Lipschitz condition with respect to y, *i.e.* there is a number $L \geqq 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqq L |y_1 - y_2|,$$

whenever $x_0 \leqq x \leqq x_0 + a$, then the whole sequence $\{y_m(x)\}$ converges uniformly on $x_0 \leqq x \leqq x_0 + a$ to the (known to be unique) solution of the original initial value problem.

The purpose of the present paper is to develop, with an eye on the numerical applications, an analogue of the Euler-Cauchy polygon method for the solution of the characteristic boundary value problem for the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, u_x, u_y),$$

(where the real-valued continuous function $f(x, y, z, p, q)$ is defined for all (x, y, z, p, q) satisfying

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad \text{and} \quad -\infty < z, p, q < +\infty).$$

The problem in question consists in the determination of a real-valued function $u(x, y)$ which satisfies the given partial differential equation on the rectangle $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$, and also satisfies the conditions

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b,$$

where $\sigma(x_0)$ $\tau(y_0)$ and $\sigma(x)$ and $\tau(y)$ are given continuously differentiable functions on the characteristics $y=y_0$ and $x=x_0$ of the given hyperbolic equation. (The treatment of this boundary value problem by successive approximations goes back to E. PICARD [5] and has been considered by various other methods by many writers since that time.) For each pair of positive integers m and n, consider the following subdivisions of the intervals

$$x_0 \leqq x \leqq x_0 + a \quad \text{and} \quad y_0 \leqq y \leqq y_0 + b,$$

$$x_0 \equiv x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} \equiv x_0 + a,$$

$$y_0 \equiv y_{0,n} < y_{1,n} < y_{2,n} < \cdots < y_{n-1,n} < y_{n,n} \equiv y_0 + b,$$

which produce a subdivision of the rectangle $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$. The miniature problem in the present method (see Section 3 for details) is as follows:

$$\frac{\partial^2 u_{mn}}{\partial x\, \partial y}(x, y) = A_{kl}, \quad \text{for } x_k \leqq x \leqq x_{k+1}, \quad y_l \leqq y \leqq y_{l+1},$$

$$u_{mn}(x, y_l) = D_{kl} + B_{kl}(x - x_k), \quad \text{for } x_k \leqq x \leqq x_{k+1},$$

$$u_{mn}(x_k, y) = D_{kl} + C_{kl}(y - y_l), \quad \text{for } y_l \leqq y \leqq y_{l+1},$$

where A_{kl}, B_{kl}, C_{kl} and D_{kl} are suitable constants, depending on the subrectangle (for simplicity in writing, x_k has been written for $x_{k,m}$ and y_l for $y_{l,n}$ in the formulation of the boundary value problem for the subrectangle). This means that on each subrectangle, the approximating function u_{mn} is bilinear in (x, y), *i.e.* it is a hyperbolic paraboloid:

$$u_{mn}(x, y) = A_{kl}(x - x_k)(y - y_l) + B_{kl}(x - x_k) + C_{kl}(y - y_l) + D_{kl}.$$

The process just described reduces in the special case of the equation $u_{xy} = f(x, y, u)$ and equal subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$ to the process given by G. ZWIRNER [24, pp. 222–223], who did not consider the more general equation treated here. Similar methods, analogous to the one described above, have been employed to prove existence theorems for the same boundary value problem by P. HARTMAN & A. WINTNER [26], R. H. MOORE [29] and R. CONTI [27], but they do not appear to be as convenient for numerical purposes as the one described above, which requires knowledge only of the tabulated values of the given functions $\sigma(x)$ and $\tau(y)$ (from which the difference quotients needed may easily be calculated) and does not require the tabulated values of the first derivatives $\sigma'(x)$ and $\tau'(y)$. Mention is also made of a different, but closely related, method, also analogous to the Euler-Cauchy polygon method, given by H. LEWY [14] (see also H. BECKERT [22]) for the solution of the initial value problem for second order quasilinear partial differential equations in two independent variables, which appears to require more differentiability assumptions than the present method.

The statement of the known main results and their connection with the existing literature is given in Section 2. Section 3 contains the precise description of the analogue of the Euler-Cauchy polygon method and the construction of the double sequence of functions $\{u_{mn}(x, y)\}$ approximating a solution. Each function u_{mn}

is continuous, but not necessarily differentiable with respect to x and y on the rectangle $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$. Section 4 contains an inequality, termed the convergence inequality, which is used, together with a theorem of C. ARZELÀ [7, pp. 119—125] on the convergence of certain not necessarily continuous functions to continuous limit functions, in order to complete the proof of the existence of a solution in Sections 5 and 6.

§ 2. Statement of known results

Theorem 1. *If*

(1) *the real-valued function $f(x, y, z)$ is defined for all (x, y, z) such that*

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad -\infty < z < +\infty,$$

where x_0, y_0, a, b are real numbers, and $a \geqq 0$, $b \geqq 0$, and if $f(x, y, z)$ is continuous and bounded in absolute value, so that for a certain non-negative constant M one has

$$|f(x, y, z)| \leqq M$$

for all these (x, y, z);

(2) *the real-valued function $\sigma(x)$ is defined for all x such that $x_0 \leqq x \leqq x_0 + a$ and possesses a continuous first derivative $\sigma'(x)$ for all these x, while the real-valued function $\tau(y)$ is defined on the set $y_0 \leqq y \leqq y_0 + b$ and possesses a continuous first derivative $\tau'(y)$ for all these y (it being understood, of course, that $\sigma'(x_0)$, for example, denotes the right-hand derivative of σ at x_0, etc.); then*

(3) *there is at least one real-valued function $u(x, y)$ defined on the rectangle*

$$R: \quad x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b,$$

which is continuous, together with its partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x\, \partial y$ $(= \partial^2 u/\partial y\, \partial x)$ on R, satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial y\, \partial x}(x, y) = f(x, y, u(x, y)) \quad \text{for } (x, y) \text{ in } R$$

and the characteristic conditions

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

It is to be noticed that this theorem asserts the existence of at least one solution to the characteristic initial value problem under consideration, but that the uniqueness of the solution is not asserted, and is, in fact, in general not true. (See P. MONTEL [8, pp. 279—283].) One need only consider the following simple example of a characteristic problem (*cf.* P. HARTMAN & A. WINTNER [26, p. 84] and P. LEEHEY [23, p. 23]) consisting of the partial differential equation

$$\frac{\partial^2 u}{\partial x\, \partial y} = |u|^{1/2} \quad \text{for } 0 \leqq x \leqq a, \quad 0 \leqq y \leqq b,$$

and the initial conditions

$$u(x, 0) = 0 \quad \text{for } 0 \leqq x \leqq a,$$

$$u(0, y) = 0 \quad \text{for } 0 \leqq y \leqq b,$$

which has as solutions both

$$u_1(x, y) = 0,$$

and

$$u_2(x, y) = \tfrac{1}{16} x^2 y^2,$$

on the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

Theorem 2. *If*

(1) *the real-valued function $f(x, y, z, p, q)$ is defined for all (x, y, z, p, q) such that*

$$x_0 \leqq x \leqq x_0 + a, \qquad y_0 \leqq y \leqq y_0 + b, \qquad -\infty < z, p, q < +\infty,$$

and is continuous and bounded in absolute value, so that for a certain non-negative constant M one has

$$|f(x, y, z, p, q)| \leqq M$$

for all these (x, y, z, p, q), and if f satisfies a Lipschitz condition in the three arguments z, p, q (that is, there is a constant $L \geqq 0$ such that one has

$$|f(x, y, z, p, q) - f(x, y, z_1, p_1, q_1)| \leqq L|z - z_1| + L|p - p_1| + L|q - q_1|,$$

for any (z, p, q) and (z_1, p_1, q_1), whenever (x, y) lies in the rectangle

$$R: \quad x_0 \leqq x \leqq x_0 + a, \qquad y_0 \leqq y \leqq y_0 + b);$$

(2) *the real-valued function $\sigma(x)$ is defined for all x such that $x_0 \leqq x \leqq x_0 + a$ and possesses a continuous first derivative $\sigma'(x)$ for all these x, while the real-valued function $\tau(y)$ is defined for all y such that $y_0 \leqq y \leqq y_0 + b$ and possesses a continuous first derivative for all these y; then*

(3) *there is one and only one real-valued function $u(x, y)$ defined on the rectangle R, which is continuous together with its partial derivatives*

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \, \partial y} \quad \left(= \frac{\partial^2 u}{\partial y \, \partial x}\right) \text{ on } R,$$

satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x \, \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right) \quad \text{for } (x, y) \text{ in } R,$$

and the characteristic conditions

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

This second theorem does not contain the first theorem as a special case, since the function $f(x, y, z)$ of Theorem 1 is not assumed to satisfy a Lipschitz condition in the argument z. However, if in Theorem 2 the function $f(x, y, z, p, q)$ does not depend on p and q, then Theorem 2 yields the additional information that if $f(x, y, z)$ of Theorem 1 does satisfy a Lipschitz condition in the argument z, then the solution whose existence is assured by Theorem 1 is indeed unique. Theorem 2 is the classical theorem of PICARD [5] mentioned in the Introduction.

Theorem 3. *If*

(1) *the real-valued function* $f(x, y, z, p, q)$ *is defined for all* (x, y, z, p, q) *such that*

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad -\infty < z, p, q < \infty,$$

and is continuous and bounded in absolute value, so that for a certain non-negative constant M *one has*

$$|f(x, y, z, p, q)| \leqq M$$

for all these (x, y, z, p, q), *and if* f *satisfies a Lipschitz condition in the two arguments* p, q *(that is, there is a constant* $L \geqq 0$ *such that one has*

$$|f(x, y, z, p, q) - f(x, y, z, p_1, q_1)| \leqq L|p - p_1| + L|q - q_1|$$

for any (p, q) *and* (p_1, q_1) *whenever* (x, y) *lies in the rectangle*

$$R: \quad x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b,$$

and z *is any real number);*

(2) *the real-valued function* $\sigma(x)$ *is defined for all* x *such that*

$$x_0 \leqq x \leqq x_0 + a,$$

and possesses a continuous first derivative $\sigma'(x)$ *for all these* x, *while the real-valued function* $\tau(y)$ *is defined for all* y *such that*

$$y_0 \leqq y \leqq y_0 + b$$

and possesses a continuous first derivative for all these y; *then*

(3) *there is at least one real-valued function* $u(x, y)$ *defined on the rectangle* R *which is continuous together with its partial derivatives* $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x\, \partial y$ $(= \partial^2 u/\partial y\, \partial x)$ *on* R, *satisfies the partial differential equation*

$$\frac{\partial^2 u}{\partial x\, \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right) \quad \text{for } (x, y) \text{ in } R,$$

and the characteristic conditions

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

This third theorem contains the first theorem as a special case (and the same example used there is applicable here). The hypotheses made in the third theorem are such that the part of the second theorem concerning the existence of a solution follows, while the second theorem yields the additional information that if the function $f(x, y, z, p, q)$ satisfies a Lipschitz condition in (z, p, q) together, rather than just in (p, q), the solution $u(x, y)$ whose existence is asserted by the third theorem is indeed unique. Theorem 3 was first proved by P. Leehey [*23*] and P. Hartman & A. Wintner [*26*]. For more general theorems see R. Conti [*27*] and A. Alexiewicz & W. Orlicz [*30*].

§ 3. The double sequence of functions approximating a solution

Let m and n be positive integers and consider the corresponding subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$, as follows:

$$x_0 = x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} = x_0 + a,$$

$$y_0 = y_{0,n} < y_{1,n} < y_{2,n} < \cdots < y_{n-1,n} < y_{n,n} = y_0 + b.$$

These subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$ produce a subdivision of the closed rectangle R into $m \cdot n$ closed subrectangles R_{kl}^{mn}, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. The closed subrectangle R_{kl}^{mn} consists in all (x, y) of R which satisfy the inequalities

$$x_{k,m} \leqq x \leqq x_{k+1,m}, \quad y_{l,n} \leqq y \leqq y_{l+1,n}.$$

Given the functions $\sigma(x)$ and $\tau(y)$, defined on the closed intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$ respectively, a continuous function $u_{mn}(x, y)$ will be defined on the rectangle R by a recurrent process, consisting in solving, on each subrectangle R_{kl}^{mn}, a boundary value problem of the form $\partial^2 u_{mn}/\partial x \partial y =$ constant, with assigned (linear) values for u_{mn} on the two rectilinear closed intervals of the boundary of R_{kl}^{mn} which intersect at its lower left hand vertex $(x_{k,m}, y_{l,n})$. Of course, the constant involved in the partial differential equation, and also the linear boundary values, both depend on k and l (and on m and n). The fact that two adjacent rectangles, say R_{kl}^{mn} and $R_{k-1,l}^{mn}$ for instance, have a common boundary interval (since they are both *closed* subrectangles) will create no difficulty concerning the definition of the function u_{mn} for points lying on the common boundary intervals, since the specific process employed in defining u_{mn} will be such that the values assigned to u_{mn} will coincide in this situation.

Suppose, for the moment, that u_{mn} has already been defined on the subrectangle of R with lower left vertex (x_0, y_0) and upper right vertex $(x_{k,m}, y_{l,m})$, *i.e.*, the subrectangle defined by the inequalities

$$x_0 \leqq x \leqq x_{k,m}, \quad y_0 \leqq y \leqq y_{l,n},$$

where $1 \leqq k < m-1$ and $1 \leqq l < n-1$. Then the definition of the function u_{mn} will be extended to the slightly larger subrectangle defined by the inequalities

$$x_0 \leqq x \leqq x_{k+1,m}, \quad y_0 \leqq y \leqq y_{l+1,n},$$

by first defining it on the closed subrectangles

$$R_{k,0}^{mn}, \; R_{k,1}^{mn}, \; \ldots, \; R_{k,l-1}^{mn}$$

in numerical succession (*i.e.*, passing from $R_{k,0}^{mn}$ to $R_{k,1}^{mn}$, and so on); then defining it on the closed subrectangles

$$R_{0,l}^{mn}, \; R_{1,l}^{mn}, \; \ldots, \; R_{k-1,l}^{mn}$$

in numerical succession (*i.e.*, passing from $R_{0,l}^{mn}$ to $R_{1,l}^{mn}$, and so on); and finally defining it on the remaining closed subrectangle R_{kl}^{mn} in order to complete the definition of u_{mn} on the rectangle

$$x_0 \leqq x \leqq x_{k+1,m}, \quad y_0 \leqq y \leqq y_{l+1,m}.$$

(A simply drawn figure will readily make the process intuitive to the reader.) Alternatively, the function $u_{mn}(x, y)$ may first be determined on the m subrectangles in a row:

$$R^{mn}_{0,l},\ R^{mn}_{1,l},\ \ldots,\ R^{mn}_{m-1,l},$$

for the rows $l = 0, 1, 2, \ldots, n-1$ in succession. There remains only to make precise just exactly what boundary value problem, *i.e.*, what partial differential equation and what boundary conditions, is to be solved on each subrectangle R^{mn}_{kl}. This will be done by showing how the process is started in the initial subrectangle R^{mn}_{00} and how the step-by-step scheme indicated above can then be carried out, using the given data, the given functions $\sigma(x)$ and $\tau(y)$. The final result will be an explicit formula for $u_{mn}(x, y)$ at any point (x, y) of a typical subrectangle R^{mn}_{kl}.

On the rectangle R^{mn}_{00} the function u_{mn} is required to satisfy the partial differential equation (with constant right-hand side)

$$\frac{\partial^2 u_{mn}}{\partial x\,\partial y}(x, y) - f\left(x_0,\ y_0,\ \sigma(x_0),\ \frac{\sigma(x_{1m}) - \sigma(x_{0m})}{x_{1m} - x_{0m}},\ \frac{\tau(y_{1n}) - \tau(y_{0n})}{y_{1n} - y_{0n}}\right) \text{ for } (x, y) \text{ in } R^{mn}_{00},$$

subject to the boundary conditions

$$u_{mn}(x, y_0) = \sigma(x_0) + \frac{\sigma(x_{1,m}) - \sigma(x_0)}{x_{1,m} - x_0}(x - x_0) \qquad \text{for } x_0 \leqq x \leqq x_{1,m},$$

$$u_{mn}(x_0, y) = \tau(y_0) + \frac{\tau(y_{1,n}) - \tau(y_0)}{y_{1,n} - y_0}(y - y_0) \qquad \text{for } y_0 \leqq y \leqq y_{1,n}.$$

Roughly speaking, what is done in defining u_{mn} on R^{mn}_{00} is to take as boundary conditions along its left boundary edge and its lower boundary edge certain linear functions derived in a natural manner from the given functions $\tau(y)$ and $\sigma(x)$, and to use the value of u_{mn} at (x_0, y_0) and the slopes of these linear functions in determining the constant value to be assigned to $\partial^2 u_{mn}/\partial x\,\partial y$ on R^{mn}_{00}. It is clear that, the boundary value problem for u_{mn} on R^{mn}_{00} being explicitly solvable,

$$\begin{aligned} u_{mn}(x, y) = {} & u_{mn}(x_0, y_0) + \\ & + \frac{u_{mn}(x_{1,m}; y_0) - u_{mn}(x_0, y_0)}{x_{1,m} - x_{0,m}}(x - x_0) + \\ & + \frac{u_{mn}(x_0; y_{1,n}) - u_{mn}(x_0, y_0)}{y_{1,n} - y_{0,n}}(y - y_0) + \\ & + f\left(x_0,\ y_0,\ u_{mn}(x_0, y_0),\ \frac{u_{mn}(x_{1,m}; y_0) - u_{mn}(x_0, y_0)}{x_{1,m} - x_{0,m}},\right. \\ & \left. \frac{u_{mn}(x_0; y_{1,n}) - u_{mn}(x_0, y_0)}{y_{1,n} - y_{0,n}}\right)(x - x_0)(y - y_0) \end{aligned}$$

for (x, y) in R^{mn}_{00}, where, for uniformity in the writing of formulas to appear later, $u_{mn}(x_0, y_0)$ has been written instead of $\sigma(x_0)$ or $\tau(y_0)$ *etc.* It is to be noticed that u_{mn} is bilinear in (x, y) on R^{mn}_{00}, *i.e.*, it is linear in x for each fixed y and linear in y for each fixed x. (From this point of view the process of defining u_{mn} being described may be thought of as a process of bilinear interpolation, so to speak.)

Consider now the definition of u_{mn} on the rectangle R_{kl}^{mn}, it being assumed that u_{mn} is *already known* as a linear function on the left boundary edge, where $x = x_{k,m}$, and on the lower boundary edge, where $y = y_{l,n}$, of the closed rectangle R_{kl}^{mn}. Then u_{mn}, on the rectangle R_{kl}^{mn}, is required to satisfy the partial differential equation with known constant right-hand side

$$\frac{\partial^2 u_{mn}}{\partial x\,\partial y}(x,y) = f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}};$$
$$\frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr) \quad \text{for } (x,y) \text{ in } R_{kl}^{mn},$$

and to coincide with an already known linear function of y on the left boundary edge, where $x = x_{k,m}$, and with another already known linear function of x on the lower boundary edge, where $y = y_{l,n}$. It is clear that u_{mn} is bilinear in (x, y) on R_{kl}^{mn} and that

$$u_{mn}(x,y) = u_{mn}(x_{k,m};\ y_{l,n}) +$$
$$+ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}}(x - x_{k,m}) +$$
$$+ \frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}(y - y_{l,n}) +$$
$$+ f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}};$$
$$\frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr)(x - x_{k,m})(y - y_{l,n}), \text{ for } (x,y) \text{ in } R_{kl}^{mn}.$$

This last formula does not exhibit the explicit dependence of the function $u_{mn}(x, y)$ on the given functions $\sigma(x)$, $\tau(y)$. In order to obtain a formula which makes evident this explicit dependence on σ and τ, which will be essential in the convergence proofs to follow, it is convenient to use an abbreviated notation yielding more manageable formulas. For example, when considering the function u_{mn}, with m and n regarded as fixed throughout the discussion, a functional value such as

$$u_{mn}(x_{k,m};\ y_{l,n})$$

will be denoted simply by u_{kl}, and a functional value such as

$$f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,n};\ y_{l,n})}{x_{k+1,m} - x_{k,m}};$$
$$\frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr)$$

will be denoted merely by f_{kl}. Further, x_k and y_l will replace $x_{k,m}$ and $y_{l,n}$, respectively.

In this notation, the above formula for $u_{mn}(x, y)$, for (x, y) in R_{kl}^{mn}, may be rewritten

$$u_{mn}(x,y) = u_{kl} + \frac{u_{k+1,l} - u_{kl}}{x_{k+1} - x_k}(x - x_k) + \frac{u_{k,l+1} - u_{kl}}{y_{l+1} - y_l}(y - y_l) + f_{kl}\cdot(x - x_k)(y - y_l).$$

Using this abbreviated notation, one has the following formulas for u_{mn} on each of the rectangles R_{00}^{mn}, R_{10}^{mn}, R_{01}^{mn}, and R_{11}^{mn}, which are all special cases of the last formula just written for R_{kl}^{mn}, for $(k, l) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively. In the first place

$$u_{mn}(x, y) = u_{00} + \frac{u_{10} - u_{00}}{x_1 - x_0}(x - x_0) + \frac{u_{01} - u_{00}}{y_1 - y_0}(y - y_0) + f_{00}(x - x_0) \cdot (y - y_0)$$

for (x, y) in R_{00}^{mn}, that is, when $x_0 \leqq x \leqq x_1$ and $y_0 \leqq y \leqq y_1$. In the second place

$$u_{mn}(x, y) = u_{10} + \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{11} - u_{10}}{y_1 - y_0}(y - y_0) + f_{10} \cdot (x - x_1)(y - y_0),$$

for (x, y) in R_{10}^{mn}, that is, when $x_1 \leqq x \leqq x_2$ and $y_0 \leqq y \leqq y_1$. In the third place

$$u_{mn}(x, y) = u_{01} + \frac{u_{11} - u_{01}}{x_1 - x_0}(x - x_0) + \frac{u_{02} - u_{01}}{y_2 - y_1}(y - y_1) + f_{01} \cdot (x - x_0)(y - y_1),$$

for (x, y) in R_{01}^{mn}, that is, when $x_0 \leqq x \leqq x_1$ and $y_1 \leqq y \leqq y_2$. In the fourth place

$$u_{mn}(x, y) = u_{11} + \frac{u_{21} - u_{11}}{x_2 - x_1}(x - x_1) + \frac{u_{12} - u_{11}}{y_2 - y_1}(y - y_1) + f_{11} \cdot (x - x_1)(y - y_1),$$

for (x, y) in R_{11}^{mn}, that is, when $x_1 \leqq x \leqq x_2$ and $y_1 \leqq y \leqq y_2$.

The formulas for R_{01}^{mn}, R_{10}^{mn} and R_{11}^{mn} will now be rewritten so as to reveal the exact influence of the given functions $\sigma(x)$ and $\tau(y)$. From the formula for (x, y) in R_{00}^{mn} it follows that

$$u_{11} = u_{10} + u_{01} - u_{00} + f_{00} \cdot (x_1 - x_0)(y_1 - y_0).$$

Substituting this expression for u_{11} into the formulas for (x, y) in R_{10}^{mn} and R_{01}^{mn}, one obtains

$$\begin{aligned} u_{mn}(x, y) = u_{10} &+ \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{01} - u_{00}}{y_1 - y_0}(y - y_0) + \\ &+ f_{00} \cdot (x_1 - x_0)(y - y_0) + f_{10} \cdot (x - x_1)(y - y_0), \end{aligned}$$

when (x, y) is in R_{10}^{mn}, and that

$$\begin{aligned} u_{mn}(x, y) = u_{01} &+ \frac{u_{10} - u_{00}}{x_1 - x_0}(x - x_0) + \frac{u_{02} - u_{11}}{y_2 - y_1}(y - y_1) + \\ &+ f_{00} \cdot (x - x_0)(y_1 - y_0) + f_{01}(x - x_0)(y - y_1), \end{aligned}$$

when (x, y) is in R_{01}^{mn}.

Now from these last two formulas for (x, y) in R_{10}^{mn} and R_{01}^{mn} one obtains

$$u_{21} = u_{01} + u_{20} - u_{00} + f_{00}(x_1 - x_0)(y_1 - y_0) + f_{10}(x_2 - x_1)(y_1 - y_0),$$

and

$$u_{12} = u_{10} + u_{02} - u_{00} + f_{00}(x_1 - x_0)(y_1 - y_0) + f_{01}(x_1 - x_0)(y_2 - y_1);$$

these, together with the already known equation

$$u_{11} = u_{10} + u_{01} - u_{00} + f_{00} \cdot (x_1 - x_0)(y_1 - y_0),$$

can be used to rewrite the formula for (x, y) in R_{11}^{mn} as follows:

$$u_{mn}(x, y) = u_{10} + u_{01} - u_{00} + \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{02} - u_{01}}{y_2 - y_1}(y - y_1) + \\ + f_{00} \cdot (x_1 - x_0)(y_1 - y_0) + f_{10} \cdot (x - x_1)(y_1 - y_0) + \\ + f_{01} \cdot (x_1 - x_0)(y - y_1) + f_{11}(x - x_1)(y - y_1),$$

for (x, y) in R_{11}^{mn}.

From the preceding considerations, the following general formula may be obtained by a process of mathematical induction:

$$u_{mn}(x, y) = u_{k,0} + u_{0,l} - u_{00} + \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k}(x - x_k) + \frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l}(y - y_l) + \\ + \sum_{i=1}^{k} \sum_{j=1}^{l} f_{i-1,j-1}(x_i - x_{i-1})(y_j - y_{j-1}) + \sum_{j=1}^{l} f_{k,j-1}(x - x_k)(y_j - y_{j-1}) + \\ + \sum_{i=1}^{k} f_{i-1,l}(x_i - x_{i-1})(y - y_l) + f_{kl}(x - x_k)(y - y_l)$$

for (x, y) in R_{kl}^{mn}, that is, when $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. It is readily seen that by putting (k, l) equal to $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ in turn one obtains the formulas given above for R_{00}^{mn}, R_{10}^{mn}, R_{01}^{mn}, R_{11}^{mn}, respectively, as special cases.

For each pair of positive integers m and n, there has been defined a subdivision of the rectangle R into $m \cdot n$ closed subrectangles, and there has also been defined on the rectangle R a real valued continuous function $u_{mn}(x, y)$. This double sequence of continuous functions $\{u_{mn}(x, y)\}$ is equibounded in absolute value on R. For let A, B, C, D denote non-negative real constants such that

$$|\sigma(x)| \leqq A, \quad |\sigma(x) - \sigma(x^*)| \leqq C|x - x^*|,$$

whenever $x_0 \leqq x \leqq x_0 + a$, and $x_0 \leqq x^* \leqq x_0 + a$; and

$$|\tau(y)| \leqq B, \quad |\tau(y) - \tau(y^*)| \leqq D|y - y^*|,$$

whenever $y_0 \leqq y \leqq y_0 + b$, and $y_0 \leqq y^* \leqq y_0 + b$. (The existence of these constants A, B, C, D follows from the assumptions made about the functions $\sigma(x)$ and $\tau(y)$ in any of the three theorems of Section 2.) Then, given (x, y) in R, one has $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$ for some suitable pair of integers k and l, with $0 \leqq k \leqq m-1$ and $0 \leqq l \leqq n-1$. Hence

$$|u_{mn}(x, y)| \leqq |u_{k,0}| + |u_{0,l}| + |u_{00}| + \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k}\right| |x - x_k| + \left|\frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l}\right| |y - y_l| + \\ + \sum_{i=1}^{k} \sum_{j=1}^{l} M(x_i - x_{i-1})(y_j - y_{j-1}) + \sum_{j=1}^{l} M(x - x_k)(y_j - y_{j-1}) + \\ + \sum_{i=1}^{k} M(x_i - x_{i-1})(y - y_l) + M(x - x_k)(y - y_l),$$

where $M \geqq 0$ is an upper bound for the absolute value of the function f (see the hypotheses of Theorems 1 to 3). Thus, by use of the definitions of the constants A, B, C, D just given, it follows that

$$|u_{mn}(x, y)| \leqq 2A + B + Ca + Db + Mab,$$

where the numerical constant on the right-hand side does not depend on the point (x, y) of R or on the pair of positive integers (m, n). This proves that the double sequence of continuous functions $\{u_{mn}(x, y)\}$ is equibounded in absolute value on R.

Now, for each pair of positive integers (m, n), let the positive numbers α_m and β_n be defined by

$$\alpha_m = \max_{k=0,1,\ldots,m-1} (x_{k+1} - x_k)$$

and

$$\beta_n = \max_{l=0,1,\ldots,n-1} (y_{l+1} - y_l),$$

so that the product $\alpha_m \cdot \beta_n$ is certainly not less than the area of the largest subrectangle of the subdivision of R corresponding to the pair of positive integers (m, n). Under the additional restriction that

$$\lim_{m\to\infty} \alpha_m = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 0$$

(which implies, but is not implied by, the fact that the maximum area of the largest subrectangle of the $(m, n)^{\text{th}}$ subdivision of R approaches zero) it will be shown that the double sequence of continuous functions $\{u_{mn}(x, y)\}$ is an equicontinuous double sequence of functions on R. By this is meant that if $\{u_{m_r n_r}(x, y)\}$ is any *singly* infinite sequence of functions (with $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$) extracted from the double sequence $\{u_{mn}(x, y)\}$, then the set of all functions $u_{m_r n_r}$, where $r = 1, 2, 3, \ldots$, is an equicontinuous set of functions.

In order to show this, one has to find an upper bound for the absolute value of the difference $u_{mn}(\bar{x}, \bar{y}) - u_{mn}(x, y)$, where $(\bar{x}, \bar{y})$ and (x, y) are points of R. There are really four cases to consider, depending on the relative positions of the points $(\bar{x}, \bar{y})$ and (x, y) with respect to each other namely; $x \leqq \bar{x}$ and $y \leqq \bar{y}$; $\bar{x} \leqq x$ and $\bar{y} \leqq y$; $x \leqq \bar{x}$ and $\bar{y} \leqq y$; $\bar{x} \leqq x$ and $y \leqq \bar{y}$. The first two cases are essentially the same by symmetry, *i.e.* by interchanging the roles of (x, y) and $(\bar{x}, \bar{y})$, and a similar remark applies to the last two cases. Only the first case mentioned will be considered here, since the treatment in the third case is exactly analogous to it. In the first case one has $x \leqq \bar{x}$, $y \leqq \bar{y}$ and $x_k \leqq x \leqq x_{k+1}$, $y_l \leqq y \leqq y_{l+1}$, and $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$, $y_{\bar{l}} \leqq \bar{y} \leqq y_{\bar{l}+1}$ for suitable pairs of integers (k, l) and $(\bar{k}, \bar{l})$. Further, $x_k \leqq x_{\bar{k}}$, $y_l \leqq y_{\bar{l}}$ and $x_{k+1} \leqq x_{\bar{k}+1}$, $y_{l+1} \leqq y_{\bar{l}+1}$.

From the definition of the function u_{mn} it follows that

$$\begin{aligned} u_{mn}(\bar{x}, \bar{y}) - u_{mn}(x, y) = u_{\bar{k},0} - u_{k,0} + u_{0,\bar{l}} - u_{0,l} + \\ &+ \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}(\bar{x} - x_{\bar{k}}) + \frac{u_{0,\bar{l}+1} - u_{0,\bar{l}}}{y_{\bar{l}+1} - y_{\bar{l}}}(\bar{y} - y_{\bar{l}}) - \\ &- \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k}(x - x_k) - \frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l}(y - y_l) + \\ &+ \left|\sum_{i=1}^{\bar{k}} \sum_{j=1}^{\bar{l}} - \sum_{i=1}^{k} \sum_{j=1}^{l}\right| [f_{i-1,j-1}(x_i - x_{i-1})(y_j - y_{j-1})] + \\ &+ \sum_{j=1}^{\bar{l}} f_{\bar{k},j-1}(\bar{x} - x_{\bar{k}})(y_j - y_{j-1}) + \sum_{i=1}^{\bar{k}} f_{i-1,\bar{l}}(x_i - x_{i-1})(\bar{y} - y_{\bar{l}}) - \\ &- \sum_{j=1}^{l} f_{k,j-1}(x - x_k)(y_j - y_{j-1}) - \sum_{i=1}^{k} f_{i-1,l}(x_i - x_{i-1})(y - y_l) + \\ &+ f_{\bar{k}\bar{l}}(\bar{x} - x_{\bar{k}})(\bar{y} - y_{\bar{l}}) - f_{kl}(x - x_k)(y - y_l). \end{aligned}$$

Hence

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| \leqq C(x_{\bar{k}+1} - x_k) + D(y_{\bar{l}+1} - y_l) + 2C\alpha_m + 2D\beta_n + \\ + M\cdot[(x_{\bar{k}} - x_0)(y_{\bar{l}} - y_0) - (x_k - x_0)(y_l - y_1)] + 2Mb\alpha_m + 2M\alpha\beta_n + 2M\alpha_m\beta_n,$$

in terms of the constants A, B, C, D, M which were introduced earlier. However,

$$x_{\bar{k}+1} - x_k \leqq (\bar{x} + \alpha_m) - (x - \alpha_m) = (\bar{x} - x) + 2\alpha_m,$$

and similarly

$$y_{\bar{l}+1} - y_l \leqq (\bar{y} + \beta_n) - (y - \beta_n) = (\bar{y} - y) + 2\beta_n$$

while

$$\begin{aligned}(x_{\bar{k}} - x_0)(y_{\bar{l}} - y_0) &- (x_k - x_0)(y_l - y_0) \\ &\leqq (\bar{x} + \alpha_m - x_0)(\bar{y} + \beta_n - y_0) - (x - \alpha_m - x_0)(y - \beta_n - y_0) \\ &\leqq (\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0) + \\ &\quad + \alpha_m[(\bar{y} - y_0) + (y - y_0)] + \beta_n[(\bar{x} - x_0) + (x - x_0)] \\ &\leqq (\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0) + 2b\alpha_m + 2a\beta_n,\end{aligned}$$

so that finally

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| \leqq 4(C + Mb)\alpha_m + 4(D + Ma)\beta_n + \\ + C(\bar{x} - x) + D(\bar{y} - y) + 2M\alpha_m\beta_n + M[(\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0)].$$

Suppose $\varepsilon > 0$ is given. Since

$$\lim_{m\to\infty} \alpha_m = \lim_{n\to\infty} \beta_n = 0,$$

there are positive integers m_ε and n_ε such that

$$4(C + Mb)\alpha_m + 4(D + Ma)\beta_n + 2M\alpha_m\beta_n < \tfrac{1}{2}\varepsilon$$

whenever $m > m_\varepsilon$ and $n > n_\varepsilon$. Further, in view of the continuity of the functions involved, there is a number $\delta_\varepsilon > 0$, which does not depend on m and n and is such that

$$C(\bar{x} - x) + D(\bar{y} - y) + M[(\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0)] < \tfrac{1}{2}\varepsilon$$

whenever $|x - \bar{x}| < \delta_\varepsilon$ and $|y - \bar{y}| < \delta_\varepsilon$. Thus, whenever $m > m_\varepsilon$ and $n > n_\varepsilon$ and

$$|x - \bar{x}| < \delta_\varepsilon, \quad |y - \bar{y}| < \delta_\varepsilon,$$

one has

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| < \varepsilon.$$

Now, let $\{u_{m_r n_r}(x,y)\}$, where $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$, be a *singly* infinite subsequence extracted from the *double* sequence $\{u_{mn}(x,y)\}$. Given $\varepsilon > 0$, one certainly has $m_r > m_\varepsilon$ and $n_r > n_\varepsilon$ for all but a finite number of positive integers r, and hence

$$|u_{m_r n_r}(\bar{x},\bar{y}) - u_{m_r n_r}(x,y)| < \varepsilon \quad \text{whenever both} \quad |x - \bar{x}| < \delta_\varepsilon \text{ and } |y - \bar{y}| < \delta_\varepsilon.$$

Since only a *finite* number of values of r are excluded and the corresponding *finite* number of excluded functions $u_{m_r n_r}$ are continuous (hence uniformly continuous) on the rectangle R, it easily follows that the set of functions $u_{m_r n_r}$, where $r = 1, 2, 3, \ldots$, is an equicontinuous set of functions as desired.

It follows then from ASCOLI's theorem (see ASCOLI [*1*] or TONELLI [*11*]) that there is a subsequence of $\{u_{m_r n_r}(x, y)\}$ which converges uniformly on R to a continuous limit function. This information is all that is really needed to complete the proof of Theorem 1 of Section 2 (where f depends only on (x, y, z)), as can be easily seen by particularizing the considerations of the following sections, and for this reason the proof will not be carried out in detail here.

The formula for $u_{mn}(x, y)$ given above was derived by carrying out a step-by-step process such as would take place in an actual numerical solution. An alternative derivation of the formula for $u_{mn}(x, y)$ will now be given. This second derivation seems to have the advantage of leading more quickly than the step-by-step method to a formula of the desired kind for other boundary value problems as well as for the present one.

First, it will be recalled that if the function $F(x, y)$ is continuous for (x, y) in R, and the functions $G(x)$ and $H(y)$ are continuously differentiable on $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$, respectively (and $G(x_0) = H(y_0)$), then there is one and only one function $w(x, y)$ which is continuous in R, together with $\partial w/\partial x$, $\partial w/\partial y$, and $\partial^2 w/\partial x\, \partial y$ $(= \partial^2 w/\partial y\, \partial x)$ and satisfies the boundary value problem

$$\frac{\partial^2 w}{\partial x\, \partial y}(x, y) = F(x, y) \qquad \text{for } (x, y) \text{ in } R,$$

$$w(x, y_0) = G(x) \qquad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$w(x_0, y) = H(y) \qquad \text{for } y_0 \leqq y \leqq y_0 + b.$$

The function $w(x, y)$ is given by the formula

$$w(x, y) = G(x) + H(y) - w(x_0, y_0) + \int_{x_0}^{x} \int_{y_0}^{y} F(\xi, \eta)\, d\xi\, d\eta,$$

where $w(x_0, y_0) = G(x_0) = H(y_0)$.

Consider the subdivisions

$$x_0 < x_1 < x_2 < \cdots < x_m < x_0 + a,$$

$$y_0 < y_1 < y_2 < \cdots < y_n < y_0 + b,$$

which were employed in the step-by-step process leading to the equation for $u_{mn}(x, y)$. By use of this subdivision of the rectangle R, the formula for $w(x, y)$ may be rewritten as follows:

$$w(x, y) = G(x) + H(y) - w(x_0, y_0) + \sum_{i=1}^{k} \sum_{j=1}^{l} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} F(\xi, \eta)\, d\xi\, d\eta +$$

$$+ \sum_{j=1}^{l} \int_{x_k}^{x} \int_{y_{j-1}}^{y_j} F(\xi, \eta)\, d\xi\, d\eta + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_{y_l}^{y} F(\xi, \eta)\, d\xi\, d\eta + \int_{x_k}^{x} \int_{y_l}^{y} F(\xi, \eta)\, d\xi\, d\eta.$$

This rewriting of the equation for $w(x, y)$ makes no difference under the assumptions made about the functions $F(x, y)$, $G(x)$, and $H(y)$. But it makes a difference when the differentiability and continuity requirements concerning $F(x, y)$, $G(x)$ and $H(y)$ are relaxed slightly. Specifically, suppose that $F(x, y)$ is bounded in absolute value throughout R and continuous at all *interior* points of each subrectangle R_{kl}^{mn}, with possible discontinuities allowed on the boundary

of any such subrectangle. Suppose also that $G(x)$ is continuous throughout $x_0 \leq x \leq x_0 + a$ and continuously differentiable for each x interior to a subinterval (*i.e.*, such that $x_k < x < x_{k+1}$ for some k) but that the derivative of $G(x)$ need not exist for the subdivision numbers x_k. Similarly, suppose also that $H(y)$ is continuous throughout $y_0 \leq y \leq y_0 + b$ and continuously differentiable for each y interior to a subinterval (*i.e.*, such that $y_l < y < y_{l+1}$ for some l) but that the derivative of $H(y)$ need not exist for the subdivision numbers y_l. The requirement that $G(x_0) = H(y_0)$ is still retained. Under these relaxed assumptions, the rewritten formula for $w(x, y)$ shows immediately that $w(x, y)$ is continuous on R and satisfies the partial differential equation

$$\frac{\partial^2 w}{\partial x\, \partial y}(x, y) = \frac{\partial^2 w}{\partial y\, \partial x}(x, y) = F(x, y)$$

whenever (x, y) is *interior* to a subrectangle R_{kl}^{mn}. Further

$$w(x, y_0) = G(x) \quad \text{for } x_0 \leq x \leq x_0 + a,$$
$$w(x_0, y) = H(y) \quad \text{for } y_0 \leq y \leq y_0 + b.$$

This last observation and the rewritten formula for $w(x, y)$ furnish immediately the desired formula for $u_{mn}(x, y)$ upon taking $F(x, y)$, $G(x)$, and $H(y)$ to be certain suitably chosen functions. One need only take for $F(x, y)$ the following (piecewise constant) function defined on $x_0 \leq x \leq x_0 + a$, $y_0 \leq y \leq y_0 + b$, by

$$F(x, y) = f_{kl} \quad \text{for } x_k \leq x \leq x_{k+1} \quad \text{and} \quad y_l \leq y \leq y_{l+1},$$

while for $x = x_0 + a$ and $y = y_0 + b$

$$F(x_0 + a, y) = f_{m-1,l} \quad \text{for } y_l \leq y < y_{l+1},$$
$$F(x, y_0 + b) = f_{k,n-1} \quad \text{for } x_k \leq x < x_{k+1},$$
$$F(x_0 + a, y_0 + b) = f_{m-1,n-1},$$

where

$$k = 0, 1, \ldots, m-1, \quad l = 0, 1, \ldots, n-1,$$

while for $G(x)$ and $H(y)$, respectively, one takes the polygonal functions (compare the description of the Euler-Cauchy polygon method in the introduction):

$$G(x) = \begin{cases} \sigma(x_k) + \dfrac{\sigma(x_{k+1}) - \sigma(x_k)}{x_{k+1} - x_k}(x - x_k) \\ \qquad\qquad \text{for } x_k \leq x < x_{k+1}, \quad \text{and} \quad k = 0, 1, \ldots, m-1 \\ \sigma(x_0 + a) \qquad \text{for } x = x_0 + a \equiv x_m, \end{cases}$$

and

$$H(y) = \begin{cases} \tau(y_l) + \dfrac{\tau(y_{l+1}) - \tau(y_l)}{y_{l+1} - y_l}(y - y_l) \\ \qquad\qquad \text{for } y_l \leq y < y_{l+1}, \quad \text{and} \quad l = 0, 1, \ldots, n-1 \\ \tau(y_0 + b) \qquad \text{for } y = y_0 + b \equiv y_n. \end{cases}$$

In verifying this remark, it must be remembered that, in the abbreviated notation, one has for example

$$u_{00} \equiv u(x_0, y_0), \quad \sigma(x_k) \equiv u_{k,0}, \quad \tau(y_l) \equiv u_{0,l}.$$

§ 4. The convergence inequality

In order to complete the proof of the theorems of Section 2, one has to consider the two double sequences of "partial derivatives" with respect to x and y of the double sequence of approximating functions of the last section. The quotation marks enclosing the phrase "partial derivatives" are a reminder that these functions must be precisely defined on R, especially along the boundaries of the subrectangles of R, where jumps may occur. The exact definition of what is meant by "partial derivatives" will be taken up in Section 5. Since the "partial derivatives" in question are not necessarily continuous functions on R, in considering their convergence one cannot make use of the theorem of ASCOLI [*1*] on equibounded, equicontinuous sequences of functions employed in Section 3 above. Instead, appeal will be made to a theorem of ARZELÀ [*7*, pp. 119–125] asserting the convergence of certain sequences of not necessarily continuous functions to continuous limit functions. The lemma of the present section furnishes an inequality concerning finite sums which serves as a basis for the application of ARZELÀ's theorem in Section 5. The result of the lemma is termed here the "convergence inequality" because of the central role it plays in the convergence proof of Section 5. It is remarked that in the theory of the ordinary differential equation $dy/dx = f(x, y)$, an entirely similar rôle is played by another convergence inequality (see, for example, BLISS [*9*, pp. 88–89]). The proof of the inequality of the lemma below resembles that given by M. BRELOT [*18*, pp. 31–32] for an inequality occurring in the theory of the ordinary differential equation $dy/dx = f(x, y)$. Compare also the inequality employed by H. BECKERT [*22*, p. 13].

Lemma. *If*

(1) *t is a positive integer, $f_0, f_1, \ldots, f_t$ is a sequence of $t+1$ non-negative numbers, and $z_0, z_1, \ldots, z_t$ is a non-decreasing sequence of $t+1$ real numbers (so that $z_j - z_{j-1} \geqq 0$ for $j = 1, 2, \ldots, t$);*

(2) *the numbers $L \geqq 0$ and $\varepsilon \geqq 0$ are such that the inequality*

$$f_l \leqq \varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1})$$

is valid for $l = 1, 2, \ldots, t$; then

(3) *the inequality*

$$f_l \leqq \left\{ \prod_{i=1}^{t} [1 + L(z_i - z_{i-1})] \right\} \{\varepsilon + L f_0(z_i - z_0)\}$$

holds for $l = 1, 2, \ldots, t$.

Proof. It will be shown by mathematical induction that

$$\varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1}) \leqq \left\{ \prod_{i=1}^{l} [1 + L(z_i - z_{i-1})] \right\} \{\varepsilon + L f_0(z_1 - z_0)\}$$

for $l = 1, 2, \ldots, t$, which implies the desired conclusion of the lemma, since $1 + L(z_i - z_{i-1}) \geqq 1$ for $i = 1, 2, \ldots, t$ and $\varepsilon + L f_0(z_1 - z_0) \geqq 0$.

For $l = 1$ the asserted inequality follows from hypothesis (2) and the fact that $1 + L(z_1 - z_0) \geqq 1$, because

$$\varepsilon + L f_0(z_1 - z_0) \leqq \{1 + L(z_1 - z_0)\}\{\varepsilon + L f_0(z_1 - z_0)\}.$$

Now for the inductive step. Suppose that the inequality to be shown holds for a positive integer $l \leqq t-1$, then it will be shown to hold also for the integer $l+1$ in place of l. This is readily seen, because then, by the inductive hypothesis and hypothesis (2) of the lemma, one has

$$f_l \leqq \varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1}) \leqq \left\{ \prod_{i=1}^{l} \left[1 + L(z_i - z_{i-1})\right] \right\} \{\varepsilon + L f_0 (z_1 - z_0)\}$$

which, together with the equality

$$\varepsilon + L \sum_{j=1}^{l+1} f_{j-1}(z_j - z_{j-1}) = \left\{ \varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1}) \right\} + f_l \cdot L \cdot (z_{l+1} - z_l),$$

implies that

$$\varepsilon + L \sum_{j=1}^{l+1} f_{j-1}(z_j - z_{j-1}) \leqq \left\{ \prod_{i=1}^{l+1} \left[1 + L(z_i - z_{i-1})\right] \right\} \{\varepsilon + L f_0 (z_1 - z_0)\},$$

and the proof is complete.

It is of some interest, although it is not needed in the considerations that follow, to point out that the inequality contained in the lemma just proved is a finite difference analogue of an inequality due to T. H. GRONWALL [*10*, p. 293], in the continuous case. (See also G. SANSONE [*21*, vol. I, pp. 30–31].) Making suitable changes (in order to conform with the present notation) in the statement of GRONWALL's inequality, as given by SANSONE, one obtains the following result:

If $f(z)$ is a non-negative continuous function defined on the interval $z_0 \leqq z \leqq z_0 + a$ and there exist numbers $\varepsilon \geqq 0$ and $L \geqq 0$ such that

$$0 \leqq f(z) \leqq \varepsilon + \int_{z_0}^{z} L f(t)\, dt$$

for $z_0 \leqq z \leqq z_0 + a$, then

$$0 \leqq f(z) \leqq \varepsilon e^{La} \qquad \text{for } z_0 \leqq z \leqq z_0 + a.$$

In order to *compare* this last inequality with the inequality of the lemma proved here, for each positive integer t consider the following subdivision of the interval $z_0 \leqq z \leqq z_0 + a$:

$$z_0 = z_{0,t} \leqq z_{1,t} \leqq z_{2,t} \leqq \cdots \leqq z_{t-1,t} \leqq z_{t,t} = z_0 + a,$$

and suppose that the hypothesis (2) of the lemma holds, with z_j and f_j being replaced, respectively, by $z_{j,t}$ and $f(z_{j,t})$. Then the conclusion of the lemma proved reads

$$\begin{aligned} f(z_{j,t}) &\leqq \left\{ \prod_{i=1}^{t} \left[1 + L(z_{i,t} - z_{i-1,t})\right] \right\} \{\varepsilon + L f_0 (z_{1,t} - z_{0,t})\} \\ &\leqq \left\{ \prod_{i=1}^{t} e^{L(z_{i,t} - z_{i-1,t})} \right\} \{\varepsilon + L f_0 (z_{1,t} - z_0)\}; \end{aligned}$$

that is,

$$f(z_{j,t}) \leqq e^{La} \cdot \{\varepsilon + L f_0 (z_{1,t} - z_0)\},$$

whose relationship in the limit to the inequality of GRONWALL cited above,

$$f(z) \leqq \varepsilon e^{La},$$

is clear.

§ 5. The double sequence of functions approximating the partial derivatives of a solution

Consider the double sequence of approximating continuous functions u_{mn} defined in Section 3. It has been pointed out at the beginning of Section 4 that the partial derivative with respect to x of u_{mn} exists in the usual sense and is finite on R save possibly when x is equal to one of the finite set of numbers (recall the abbreviated notation introduced at the end of Section 3)

$$x_1 < x_2 < \cdots < x_{m-1},$$

where jumps may occur. (Of course, it is understood that when $x = x_0$ and $x = x_0 + a$, by the "partial derivative with respect to x" of the function u_{mn} are meant the one-sided limits

$$\lim_{\substack{\bar{x} \to x_0 \\ \bar{x} > x_0}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_0, y)}{\bar{x} - x_0}$$

and

$$\lim_{\substack{\bar{x} \to x_0 + a \\ \bar{x} < x_0 + a}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_0 + a, y)}{\bar{x} - (x_0 + a)},$$

respectively.) A similar statement applies to the derivative with respect to y of the function u_{mn}, the possible jumps now occurring when y is equal to one of the finite set of numbers

$$y_1 < y_2 < \cdots < y_{n-1},$$

a corresponding agreement being made about the "partial derivatives with respect to y" of the function u_{mn} when $y = y_0$ and $y = y_0 + b$. For reasons of symmetry, it is clear that one may restrict attention to the x derivative, similar considerations being applicable in the case of the y derivative. Intuitively speaking, it will now be shown, using the lemma of Section 4, that the magnitude of the jumps in $\partial u_{mn}/\partial x$ can be made arbitrarily small by choosing both m and n sufficiently large.

First, consider the function u_{mn} on the closed subrectangle R_{kl}^{mn}, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. By its very construction, the function u_{mn} is bilinear in x and y on the subrectangle R_{kl}^{mn}. In view of the formula for u_{mn} given in Section 3, when (x, y) is a point of the rectangle R_{kl}^{mn} which is *not* on its closed left and right-hand rectilinear boundary intervals (*i.e.*, when the point (x, y) satisfies the inequalities $x_k < x < x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$), then

$$\frac{\partial u_{mn}}{\partial x}(x, y) = \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} + \sum_{j=1}^{l} f_{k,j-1}(y - y_{j-1}) + f_{kl}(y - y_l).$$

On the other hand, when the point (x, y) is on the closed left-hand rectilinear boundary interval

$$x = x_k, \quad y_l \leqq y \leqq y_{l+1},$$

then the right-hand x derivative

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) = \lim_{\bar{x} \to x_k} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_k, y)}{\bar{x} - x_k},$$

where $x_k < \bar{x} \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$, exists and is a linear function of y. Similarly, when the point (x, y) is on the closed right-hand rectilinear boundary interval

$$x = x_{k+1}, \qquad y_l \leqq y \leqq y_{l+1},$$

then the left-hand x derivative

$$\frac{\partial^- u_{mn}}{\partial x}(x_{k+1,y}) = \lim_{\bar{x} \to x_{k+1}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_{k+1}, y)}{\bar{x} - x_{k+1}},$$

where $x_k \leqq \bar{x} < x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$ exists and is a linear function of y. It is to be noticed that the "partial derivative with respect to x" is constant on R_{kl}^{mn} for each fixed y; that is, for each y such that $y_l \leqq y \leqq y_{l+1}$ one has

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) = \frac{\partial u_{mn}}{\partial x}(x, y) = \frac{\partial^- u_{mn}}{\partial x}(x_{k+1,y})$$

for all x satisfying $x_k < x < x_{k+1}$.

The maximum absolute value of the difference between the values of the "partial derivative with respect to x" of u_{mn} on two subrectangles R_{kl}^{mn} and $R_{\bar{k}l}^{mn}$ at the same y level will now be estimated by use of the lemma of Section 4. Suppose, for definiteness, that $\bar{k} \geq k$. For the rectangle $R_{\bar{k}l}^{mn}$ there are formulas for $\partial u_{mn}/\partial x$, *etc.*, similar to those just derived for R_{kl}^{mn}, which need not be recorded here explicitly. One also has that for each y such that $y_l \leqq y \leqq y_{l+1}$ the equality

$$\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y) = \frac{\partial u_{mn}}{\partial x}(x, y) = \frac{\partial^- u_{mn}}{\partial x}(x_{\bar{k}+1}, y)$$

holds for all x satisfying $x_{\bar{k}} < x < x_{\bar{k}+1}$. Consequently, the problem of estimating the maximum absolute value of the difference between the values of the "partial derivative with respect to x" of u_{mn} on the two subrectangles R_{kl}^{mn} and $R_{\bar{k}l}^{mn}$ reduces simply to the estimation of maximum absolute value of the difference of the two functions of y,

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) \quad \text{and} \quad \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y),$$

which are *linear* functions of y on the interval $y_l \leqq y \leqq y_{l+1}$. In view of the linearity of the two functions involved, the desired maximum absolute value of their difference,

$$\max_{\substack{y_l \leqq y \leqq y_{l+1} \\ y_l \leqq \bar{y} \leqq y_{l+1}}} \left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, \bar{y}) \right|$$

is just equal to the maximum of the four numbers

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) \right|.$$

It is to be noticed that only the estimation of the first two of these numbers requires special attention, since it will turn out that the last two can be made arbitrarily small whenever the difference $y_{l-1} - y_l$ is chosen sufficiently small, the reason for this being the continuity of $\partial^+ u_{mn}/\partial x$ with respect to y for each fixed x. For example,

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|$$
$$\leqq \left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1})\right| + \left|\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|.$$

Further, since the first number is obtainable from the second merely by replacing l by $l+1$, all that remains is to estimate, for each pair of fixed integers $\bar{k} \geqq k$, the $n+1$ numbers

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|,$$

where $l = 0, 1, \ldots, n$. For $l = 0$ this absolute value can be made arbitrarily small, and the lemma of Section 4 will now be used in showing that the absolute values for $l = 1, \ldots, n$ can also be made arbitrarily small.

Now, from the definition of $\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l)$ and $\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)$ (recall, for example, that $u_{mn}(x, y_l)$, for $x_k \leqq x \leqq x_{k+1}$, is a *linear* function of the single variable x) together with the previous formula for $\partial^+ u_{mn}/\partial x$ obtained in this section, it follows that (recall that, for example, $u_{mn}(x_{k+1}, y_l) \equiv u_{k+1,l}$)

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) = \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}$$
$$= \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} + \sum_{j=1}^{l} (f_{k,j-1} - f_{\bar{k},j-1})(y_j - y_{j-1}) \quad \text{for } l = 1, \ldots, n.$$

The proof that the absolute value of this last difference can be made arbitrarily small provided that m and n are chosen sufficiently large will now be completed, at first under the hypothesis required of the function in Theorem 2, *i.e.*, that f satisfies a Lipschitz condition in all three of its last arguments z, p, q (see hypothesis (1) of Theorem 2). The argument will be carried out first in this case because it is somewhat simpler than the corresponding argument when f satisfies a Lipschitz condition only in its last two arguments p, q (see hypothesis (1) of Theorem 3). *It will also be supposed at first*, again for the sake of simplicity in writing, that the function $f(x, y, z, p, q)$ does not depend on x and y, that is $f = f(z, p, q)$.

Accordingly, under the hypothesis (1) of Theorem 2, one has that (recall the description of the abbreviated notation $f_{k,j-1}$ introduced in Section 3):

$$|f_{k,j-1} - f_{\bar{k},j-1}| \leqq L\left\{|u_{k,j-1} - u_{\bar{k},j-1}| + \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \right.$$
$$\left. + \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right|\right\},$$

and consequently

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar k}, y_l)\right| = \left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,l} - u_{\bar k,l}}{x_{\bar k+1} - x_{\bar k}}\right|$$

$$\leq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,0} - u_{\bar k,0}}{x_{\bar k+1} - x_{\bar k}}\right|$$

$$+ L \cdot \sum_{j=1}^{l} \left\{ \left|u_{k,j-1} - u_{\bar k,j-1}\right| + \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,j-1} - u_{\bar k,j-1}}{x_{\bar k+1} - x_{\bar k}}\right| + \right.$$

$$\left. + \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar k,j} - u_{\bar k,j-1}}{y_j - y_{j-1}}\right| \right\} (y_j - y_{j-1}).$$

The term in the last inequality which involves the difference quotients with respect to y requires special attention. Consider the function u_{mn} on the rectangle $R^{mn}_{k,j-1}$. From the formula for u_{mn} given in Section 3 it follows that

$$u_{k,j-1} = u_{k,0} + u_{0,j-1} - u_{00} + \sum_{i=1}^{k} \sum_{J=1}^{j-1} f_{i-1,J-1}(x_i - x_{i-1})(y_J - y_{J-1}),$$

and

$$u_{k,j} = u_{k,0} + u_{0,j} - u_{00} + \sum_{i=1}^{k} \sum_{J=1}^{j-1} f_{i-1,J-1}(x_i - x_{i-1})(y_J - y_{J-1}) +$$

$$+ \sum_{i=1}^{k} f_{i-1,j-1}(x_i - x_{i-1})(y_j - y_{j-1});$$

hence

$$\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} = \frac{u_{0,j} - u_{0,j-1}}{y_j - y_{j-1}} + \sum_{i=1}^{k} f_{i-1,j-1}(x_i - x_{i-1}).$$

Similarly

$$\frac{u_{\bar k,j} - u_{\bar k,j-1}}{y_j - y_{j-1}} = \frac{u_{0,j} - u_{0,j-1}}{y_j - y_{j-1}} + \sum_{i=1}^{\bar k} f_{i-1,j-1}(x_i - x_{i-1}),$$

and thus

$$\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar k,j} - u_{\bar k,j-1}}{y_j - y_{j-1}} = -\sum_{i=k+1}^{\bar k} f_{i-1,j-1}(x_i - x_{i-1}).$$

Further, in view of this

$$\sum_{j=1}^{l} \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar k,j} - u_{\bar k,j-1}}{y_j - y_{j-1}}\right| (y_j - y_{j-1}) = \sum_{j=1}^{l} \left|\sum_{i=k+1}^{\bar k} f_{i-1,j-1}(x_i - x_{i-1})\right| (y_j - y_{j-1})$$

$$\leq M\left[\sum_{i=k+1}^{\bar k}(x_i - x_{i-1})\right]\left[\sum_{j=1}^{l}(y_j - y_{j-1})\right] \leq M\left[\sum_{i=k+1}^{\bar k}(x_i - x_{i-1})\right] \cdot b = M b (x_{\bar k} - x_k).$$

The inequality for

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar k}, y_l)\right|$$

may now be rewritten in the form

$$\left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,l} - u_{\bar k,l}}{x_{\bar k+1} - x_{\bar k}}\right| \leq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,0} - u_{\bar k,0}}{x_{\bar k+1} - x_{\bar k}}\right| +$$

$$+ L \sum_{j=1}^{l} \left|u_{k,j-1} - u_{\bar k,j-1}\right| (y_j - y_{j-1}) + M b (x_{\bar k} - x_k) +$$

$$+ L \cdot \sum_{j=1}^{l} \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar k+1,j-1} - u_{\bar k,j-1}}{x_{\bar k+1} - x_{\bar k}}\right| (y_j - y_{j-1}) \quad \text{for } l = 1, 2, \ldots, n.$$

This last inequality is precisely of the same type as that of hypothesis (2) of the lemma in Section 4, upon identifying, in particular, t with n, the f_j and z_j occurring there (for $j = 0, 1, \ldots, n$) with the present

$$\left| \frac{u_{k+1,j} - u_{k,j}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j} - u_{\bar{k},j}}{x_{\bar{k}+1} - x_{\bar{k}}} \right|$$

and y_j, respectively, and the ε of the lemma with

$$\max_{0 \leqq k < \bar{k} \leqq m-1} \left\{ \left| \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| + \right.$$
$$\left. + L \sum_{j=1}^{l} \left| u_{k,j-1} - u_{\bar{k},j-1} \right| (y_j - y_{j-1}) + M b (x_{\bar{k}} - x_k) \right\},$$

which, as will now be shown, can be made arbitrarily small merely by choosing m and n sufficiently large and $|x_k - x_{\bar{k}}|$ sufficiently small (in view of the assumed continuity of the derivative $\sigma'(x)$ and the equicontinuity of any subsequence $\{u_{m_r n_r}(x, y)\}$, with $\lim\limits_{r \to \infty} m_r = \lim\limits_{r \to \infty} n_r = \infty$, which was shown in Section 3). In verifying this, one can use the mean value theorem of the differential calculus, since for $k = 1, \ldots, m-1$

$$\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} = \sigma'(x_k^*) - \sigma'(x_{\bar{k}}^*),$$

where $x_k < x_k^* < x_{k+1}$ and $x_{\bar{k}} < x_{\bar{k}}^* < x_{\bar{k}+1}$. Let $\varepsilon > 0$ be given, then there exist (see Section 3) positive integers m_ε and n_ε and a positive number δ_ε such that whenever $m_r > m_\varepsilon$, $n_r > n_\varepsilon$ and $|x_k - x_{\bar{k}}| < \delta_\varepsilon$ one has

$$\max_{0 \leqq k < \bar{k} \leqq m-1} \left\{ L \sum_{j=1}^{l} \left| u_{k,j-1} - u_{\bar{k},j-1} \right| (y_j - y_{j-1}) + M b (x_{\bar{k}} - x_k) \right\} < \tfrac{1}{2}\varepsilon,$$

where $u(x, y)$ is written for $u_{m_r n_r}(x, y)$. Also, in view of the *uniform* continuity of the function $\sigma'(x)$ on the interval $x_0 \leqq x \leqq x_0 + a$, it follows that

$$\left| \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| < \frac{1}{2}\varepsilon.$$

Consequently, from the conclusion of the lemma of Section 4 it follows that

$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} \right|$$
$$\leqq \left\{ \prod_{i=1}^{n} \left[1 + L(y_i - y_{i-1})\right] \right\} \left\{ \varepsilon + L\,\varepsilon\,(y_1 - y_0) \right\} \leqq \left\{ \prod_{i=1}^{n} e^{L(y_i - y_{i-1})} \right\} \left\{ \varepsilon + L\,\varepsilon\,(y_1 - y_0) \right\}$$
$$= e^{Lb} \left\{ \varepsilon + L\,\varepsilon\,(y_1 - y_0) \right\} \quad \text{for } l = 1, 2, \ldots, n.$$

(It should be noticed that the last inequality also holds for $l = 0$.)

The last inequality has been obtained under the *two* assumptions that the function f satisfies a Lipschitz condition in all three variables z, p, q (hypothesis (1) of Theorem 2) *and* that f does not depend explicitly on x and y; that is, $f \equiv f(z, p, q)$. The derivation of a similar inequality, in the case when $f \equiv f(x, y, z, p, q)$ satisfies *only* a Lipschitz condition in the two variables p, q (hypothesis (1) of Theorem 3) will now be sketched. As in the previous case, the initial step, where the Lipschitz condition is applied, is in estimating the absolute value

of the difference $f_{k,j-1} - f_{\bar{k},j-1}$. This can now be done as follows by adding and subtracting the number

$$f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)$$

to the difference in question. One obtains

$$\begin{aligned} f_{k,j-1} - f_{\bar{k},j-1} = f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \\ - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{1-j}}\right) + \\ + f\left(x_k;\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \\ - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}};\ \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right). \end{aligned}$$

Using this and the Lipschitz condition with respect to p and q, one has

$$\begin{aligned} &\left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| \\ &\leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \sum_{j=1}^{l} |f_{k,j-1} - f_{\bar{k},j-1}|\,(y_j - y_{j-1}) \\ &\leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \\ &\quad + \sum_{j=1}^{l} \left| f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \right. \\ &\quad \left. - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)\right| (y_j - y_{j-1}) + \\ &\quad + L \sum_{j=1}^{l} \left\{ \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \right. \\ &\quad \left. + \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right| \right\} (y_j - y_{j-1}). \end{aligned}$$

The term in the last summation involving the explicit difference quotients with respect to y may be handled exactly as before, yielding the same result:

$$\sum_{j=1}^{l} \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right| (y_j - y_{j-1}) \leqq M\,b\,(x_{\bar{k}} - x_k).$$

Thus, one has finally

$$\begin{aligned} &\left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| \leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \\ &\quad + \sum_{j=1}^{l} \left| f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \right. \\ &\quad \left. - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)\right| (y_j - y_{j-1}) + \\ &\quad + M\,b\,(x_{\bar{k}} - x_k) + L \sum_{j=1}^{l} \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| (y_j - y_{j-1}) \end{aligned}$$

for $l = 1, 2, \ldots, n$.

This inequality is again precisely of the same type as that of hypothesis (2) of the lemma in Section 4, upon identifying, in particular, t with n, the f_j and z_j occurring there (for $j = 0, 1, \ldots, n$) with the present

$$\left| \frac{u_{k+1,j} - u_{k,j}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j} - u_{\bar{k},j}}{x_{\bar{k}+1} - x_{\bar{k}}} \right|$$

and y_j, respectively, and the ε of the lemma with

$$\max_{0<k<\bar{k}\leq m-1} \left\{ \left| \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| + \right.$$
$$+ \sum_{j=1}^{l} \left| f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \right.$$
$$\left. \left. - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) \right| (y_j - y_{j-1}) + M\, b\, (x_{\bar{k}} - x_k) \right\},$$

which (as will now be indicated, without entering into the detailed argument) can be made arbitrarily small (*i.e.*, less than any positive number given in advance) merely by choosing m and n sufficiently large and $|x_k - x_{\bar{k}}|$ sufficiently small. In showing this, use is made of the assumed continuity of the derivative $\sigma'(x)$; of the equicontinuity of any subsequence $\{u_{m_r n_r}(x, y)\}$ with $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$, which was shown in Section 3; and of the *uniform* continuity of the function $f(x, y, z, p, q)$ on any closed and bounded set of points (x, y, z, p, q) satisfying

$$(x, y) \text{ in } R, \quad -Z \leqq z \leqq Z, \quad -P \leqq p \leqq P, \quad -Q \leqq q \leqq Q,$$

with Z, P, Q positive numbers. Notice that it can readily be seen, from the definition of u_{mn} and of the difference quotients involved, that there exist positive numbers Z, P, Q such that

$$|u_{k,l}| \leqq Z,$$
$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} \right| \leqq P \quad \text{for } k = 0, 1, \ldots, m-1 \quad \text{and} \quad l = 0, 1, \ldots, n,$$
$$\left| \frac{u_{k,l+1} - u_{k,l}}{y_{j+1} - y_j} \right| \leqq Q \quad \text{for } k = 0, 1, \ldots, m \quad \text{and} \quad l = 0, 1, \ldots, n-1$$

and for *any* pair of positive integers m and n, where one uses the abbreviated notation, $u_{kl} \equiv u_{mn}(x_k, y_l)$, *etc.* In particular, since

$$\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} = \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} + \sum_{j=1}^{l} f_{k,j-1}(y_j - y_{j-1}),$$

one may choose

$$P = C + M b,$$

in terms of the constants C, M and b of Sections 2 and 3. This being granted, one obtains exactly as before, by an application of the lemma of Section 4, that if $\varepsilon > 0$ is given, then there exist positive integers m_ε and n_ε and a positive number δ_ε such that whenever $m_r > m_\varepsilon$ and $n_r > n_\varepsilon$, and $|x_k - x_{\bar{k}}| < \delta_\varepsilon$ then

$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| \leqq e^{Lb}\{\varepsilon + L\,\varepsilon\,(y_1 - y_0)\} \quad \text{for } l = 0, 1, 2, \ldots, n.$$

It is now time to define the double sequence of functions $\{p_{mn}(x, y)\}$ corresponding to the double sequence $\{u_{mn}(x, y)\}$ of Section 3. The double sequence $\{p_{mn}(x, y)\}$ will be, roughly speaking, a sequence of functions approximating the partial derivative with respect to x of a solution. In view of the possibility of jumps in $\partial u_{mn}/\partial x$, the function $p_{mn}(x, y)$ has to be defined carefully in R, to make sure it is single-valued. For each pair of positive integers m and n, the function p_{mn} is defined as follows, for points (x, y) in the closed rectangle R:

$$p_{mn}(x,y) = \begin{cases} \dfrac{\partial u_{mn}}{\partial x}(x,y) & \text{whenever } x_k < x < x_{k+1} \text{ for some } k = 0, 1, \ldots, m-1, \\ \dfrac{\partial^+ u_{mn}}{\partial x}(x,y) & \text{whenever } x = x_k \text{ for some } k = 0, 1, \ldots, m-1, \\ \dfrac{\partial^- u_{mn}}{\partial x}(x,y) & \text{whenever } x = x_m = x_0 + b. \end{cases}$$

The function p_{mn} possibly has jump discontinuities only when $x = x_1, \ldots, x_{m-1}$ and is continuous in the two independent variables x and y at all other points of R.

This double sequence of functions $\{p_{mn}(x, y)\}$, as may be readily seen from the formulas given for $\partial u_{mn}/\partial x$, $\partial^+ u_{mn}/\partial x$ and $\partial^- u_{mn}/\partial x$ given earlier in this section, is equibounded in absolute value on R. That is to say, there is a positive number P, which is independent of m, n and of (x, y), such that

$$|p_{mn}(x, y)| \leqq P$$

for any positive integers m and n, and any point (x, y) of R.

Let $\{p_{m_s n_s}\}$ denote any *singly* infinite subsequence of functions (with $\lim_{s\to\infty} m_s = \lim_{s\to\infty} n_s = \infty$) extracted from the double sequence $\{p_{mn}(x, y)\}$. Let $\varepsilon > 0$. From the preceding considerations it follows that there exist positive integers m_ε and n_ε and a number $\delta_\varepsilon > 0$ such that whenever $(\bar{x}, \bar{y})$ and (x, y) are points of R satisfying

$$|x - \bar{x}| < \delta_\varepsilon, \quad |y - \bar{y}| < \delta_\varepsilon,$$

and $m_s > m_\varepsilon$, $n_s > n_\varepsilon$, then

$$|p_{m_s n_s}(\bar{x}, \bar{y}) - p_{m_s n_s}(x, y)| < \varepsilon.$$

(In ARZELÀ's terminology [7, p. 119], the subsequence of $\{p_{m_s n_s}(x, y)\}$ for which $m_s > m_\varepsilon$ and $n_s > n_\varepsilon$ is equioscillating by less than ε. This can be proved by an argument similar to that used in Section 3 in showing that the sequence $\{u_{m_r n_r}(x, y)\}$ is equicontinuous. There are again four cases to consider, depending on the relative positions of the points $(\bar{x}, \bar{y})$ and (x, y) with respect to each other. As in Section 3, only the case when $x \leqq \bar{x}$ and $y \leqq \bar{y}$ need be considered in detail. Here one has $x_k \leqq x \leqq x_{k+1}$; $y_l \leqq y \leqq y_{l+1}$; and $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$; $y_{\bar{l}} \leqq \bar{y} \leqq y_{\bar{l}+1}$ for suitable pairs of integers (k, l) and $(\bar{k}, \bar{l})$. Further $x_k \leqq x_{\bar{k}}$; $y_l \leqq y_{\bar{l}}$ and $x_{k+1} \leqq x_{\bar{k}+1}$; $y_{l+1} \leqq y_{\bar{l}+1}$. The inequalities deduced earlier in this section for

$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} \right|$$

may then readily be employed to obtain the desired result, the details being as follows. Now

$$p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y) = [p_{mn}(\bar{x},\bar{y}) - p_{mn}(\bar{x},y)] + [p_{mn}(\bar{x},y) - p_{mn}(x,y)],$$

where the point $(\bar{x}, y)$ is in the subrectangle $R_{\bar{k}l}^{mn}$ because $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$ and $y_l \leqq y \leqq y_{l+1}$. This, together with the definition of the function p_{mn}, implies the inequality

$$\begin{aligned}
|p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y)| &\leqq |p_{mn}(\bar{x},\bar{y}) - p_{mn}(\bar{x},y)| + |p_{mn}(\bar{x},y) - p_{mn}(x,y)| \\
&= \left|\left\{\frac{u_{\bar{k}+1,\bar{l}} - u_{\bar{k},\bar{l}}}{x_{\bar{k}+1} - x_{\bar{k}}} + (\bar{y} - y_{\bar{l}}) f_{\bar{k}\bar{l}}\right\} - \left\{\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} + (y - y_l) f_{\bar{k}l}\right\}\right| + \\
&\quad + \left|\left\{\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} + (y - y_l) f_{\bar{k}l}\right\} - \left\{\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} + (y - y_l) f_{kl}\right\}\right| \\
&\leqq \left|\frac{u_{\bar{k}+1,\bar{l}} - u_{\bar{k},\bar{l}}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \left|\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k}\right| + \\
&\quad + |(y - y_{\bar{l}}) f_{\bar{k}\bar{l}}| + 2|(y - y_l) f_{\bar{k}l}| + |(y - y_l) f_{kl}| \\
&\leqq \left|\sum_{j=l+1}^{\bar{l}} f_{\bar{k},j-1}(y_j - y_{j-1})\right| + \left|\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k}\right| + 4M\beta_n,
\end{aligned}$$

in case $x_k \leqq x \leqq x_{k+1}$ for some integer $k = 0, 1, \ldots, m-1$ (this only excludes $x = x_0 + a$, which will be treated separately below) and the integer $\bar{k}$ is chosen (if possible) so that $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$, with $\bar{k} = 0, 1, \ldots, m-1$ (if $\bar{x} = x_0 + a$, which is seemingly excluded at first, the inequality just written still continues to hold, but with $\bar{k}$ replaced by $m-1$). If $x = x_0 + a$, a case definitely excluded above, then one must necessarily have $\bar{x} = x_0 + a$ $(= x_m)$ too, and then

$$\begin{aligned}
&|p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y)| \\
&\quad = \left|\left\{\frac{u_{m,\bar{l}} - u_{m-1,\bar{l}}}{x_m - x_{m-1}} + (\bar{y} - y_{\bar{l}}) f_{m-1,\bar{l}}\right\} - \left\{\frac{u_{m,l} - u_{m-1,l}}{x_m - x_{m-1}} + (y - y_l) f_{m-1,l}\right\}\right| \\
&\quad \leqq \left|\sum_{j=l+1}^{\bar{l}} f_{m-1,j-1}(y_j - y_{j-1})\right| + 2M\beta_n.
\end{aligned}$$

These inequalities now readily furnish the desired "equioscillation" property of the singly infinite sequence $\{p_{m_s n_s}\}$.

Since the sequence of functions $\{p_{m_s n_s}\}$ is equibounded in absolute value, and since for each $\varepsilon > 0$ there are positive integers m_ε and n_ε and a number $\delta_\varepsilon > 0$ such that for all points $(\bar{x},\bar{y})$ and (x, y) of R satisfying $|x - \bar{x}| < \delta_\varepsilon$, $|y - \bar{y}| < \delta_\varepsilon$ and for all m_s and n_s satisfying $m_s > m_\varepsilon$, $n_s > n_\varepsilon$ one has

$$|p_{m_s n_s}(\bar{x},\bar{y}) - p_{m_s n_s}(x,y)| < \varepsilon,$$

it follows from a theorem of ARZELÀ [7, pp. 119—125] that there is a continuous function $p(x, y)$ defined on R and a subsequence of the sequence $\{p_{m_s n_s}\}$ which converges *uniformly* to the continuous function $p(x, y)$ on R. For a proof of this particular result needed here, carried out under the equivalent hypothesis that the given sequence of functions has zero "Grenzschwankung" (see CARATHÉODORY [*17*, p. 3] for the definition of this term), reference is made to H. BECKERT [*22*, pp. 24—27].

For reasons of symmetry, without further discussion it is clear how the double sequence $\{q_{mn}(x, y)\}$, which approximates the y derivative of a solution is defined. It is also clear that there is a positive number Q, which is independent of m, n and of (x, y) such that

$$|q_{mn}(x, y)| \leqq Q$$

for any positive integers m and n and any point (x, y) of R. Let $\{q_{m_t n_t}\}$ denote any *singly* infinite subsequence of functions with $\lim_{t\to\infty} m_t = \lim_{t\to\infty} n_t = \infty$ extracted from the double sequence of functions $\{q_{mn}(x, y)\}$. Again, by ARZELÀ's theorem, one concludes that there is a *continuous* function $q(x, y)$ defined on R and a subsequence of the sequence $\{q_{m_t n_t}(x, y)\}$ which converges *uniformly* to $q(x, y)$ on R.

§ 6. The existence of a solution

Consider the double sequences of functions $\{u_{mn}(x, y)\}$, $\{p_{mn}(x, y)\}$, and $\{q_{mn}(x, y)\}$. In Section 5 it was pointed out that there exist positive numbers Z, P, and Q such that for any positive integers m, n and any (x, y) in R, one has

$$|u_{mn}(x, y)| \leq Z, \quad |p_{mn}(x, y)| \leq P, \quad |q_{mn}(x, y)| \leq Q.$$

It is remarked, since use will be made of this fact immediately, that the continuous function $f(x, y, z, p, q)$ is *uniformly* continuous in (x, y, z, p, q) on the closed and bounded five dimensional set of points defined by

$$x_0 \leqq x \leqq x_0 + a; \quad y_0 \leqq y \leqq y_0 + b, \quad |z| \leqq Z, \quad |p| \leqq P, \quad |q| \leqq Q.$$

That is, given $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ (which may be chosen to be less than ε, for later convenience) such that whenever $(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)$ satisfy the inequalities

$$x_0 \leqq \bar{x}_i \leqq x_0 + a, \quad y_0 \leqq \bar{y}_i \leqq y_0 + a, \quad |\bar{z}_i| \leqq Z, \quad |\bar{p}_i| \leqq P, \quad |\bar{q}_i| \leqq Q \quad \text{for } i = 1, 2$$

and

$$|\bar{x}_1 - \bar{x}_2| < \delta_\varepsilon, \quad |\bar{y}_1 - \bar{y}_2| < \delta_\varepsilon, \quad |\bar{z}_1 - \bar{z}_2| < \delta_\varepsilon, \quad |\bar{p}_1 - \bar{p}_2| < \delta_\varepsilon, \quad |\bar{q}_1 - \bar{q}_2| < \delta_\varepsilon,$$

then

$$|f(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1) - f(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)| < \varepsilon.$$

Let $\{u_{m_r n_r}(x, y)\}$, where $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$, be a singly infinite subsequence of the double sequence $\{u_{mn}(x, y)\}$, and suppose further that (see Section 3)

$$\lim_{r\to\infty} u_{m_r n_r}(x, y) = u(x, y),$$

where the convergence to the continuous function $u(x, y)$ holds over the rectangle R. From Section 5, it follows that the corresponding subsequence $\{p_{m_r n_r}(x, y)\}$ itself contains a subsequence which converges uniformly on R to a continuous function $p(x, y)$. For simplicity, suppose the subscripts have been chosen so that the subsequence $\{p_{m_r n_r}\}$ itself converges uniformly on R to $p(x, y)$. Making

a similar agreement about subscripts, it may also be supposed that the corresponding subsequence $\{q_{m_r n_r}(x, y)\}$ itself converges uniformly on R to a continuous function $q(x, y)$. Summarizing, one concludes that

$$\lim_{r\to\infty} u_{m_r n_r}(x, y) = u(x, y),$$

$$\lim_{r\to\infty} p_{m_r n_r}(x, y) = p(x, y),$$

$$\lim_{r\to\infty} q_{m_r n_r}(x, y) = q(x, y),$$

the convergence to the continuous functions u, p, q being uniform on R. It will now be shown that the function $u(x, y)$ is a solution of the boundary value problem under study.

In view of the above mentioned uniform continuity of f on a certain closed and bounded five-dimensional set of points, it follows that

$$\lim_{r\to\infty} f\big(x, y, u_{m_r n_r}(x, y), p_{m_r n_r}(x, y), q_{m_r n_r}(x, y)\big) = f\big(x, y, u(x, y), p(x, y), q(x, y)\big),$$

the convergence to the continuous limit function being again uniform on R. Furthermore, since the limit function

$$f\big(x, y, u(x, y), p(x, y), q(x, y)\big)$$

is continuous on R, the following Riemann integrals exist for all (x, y) in R:

$$\int_{x_0}^{x}\int_{y_0}^{y} f\big(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta)\big)\, d\xi\, d\eta,$$

$$\int_{x_0}^{x} f\big(\xi, y, u(\xi, y), p(\xi, y), q(\xi, y)\big)\, d\xi,$$

$$\int_{y_0}^{y} f\big(x, \eta, u(x, \eta), p(x, \eta), q(x, \eta)\big)\, d\eta;$$

the order of integration with respect to ξ and η may be interchanged in the double integral without altering its value. All this information will now be used in order to show that the function p is precisely the x derivative of the function u and that the function q is precisely the y derivative of the function u.

Let $\varepsilon > 0$, and let $\delta_\varepsilon > 0$ be such that $\varepsilon > \delta_\varepsilon > 0$ (the restriction $\varepsilon > \delta_\varepsilon$ is made for later convenience) and that also

$$|f(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1) - f(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)| < \varepsilon$$

whenever the points $(\bar{x}_i, \bar{y}_i, \bar{z}_i, \bar{p}_i, \bar{q}_i)$ satisfy both

$$x_0 \leq \bar{x}_i \leq x_0 + a, \quad y_0 \leq \bar{y}_i \leq y_0 + a, \quad |\bar{z}_i| \leq Z, \quad |\bar{p}_i| \leq P, \quad |\bar{q}_i| \leq Q,$$

and

$$|\bar{x}_1 - \bar{x}_2| < \delta_\varepsilon, \quad |\bar{y}_1 - \bar{y}_2| < \delta_\varepsilon, \quad |\bar{z}_1 - \bar{z}_2| < \delta_\varepsilon, \quad |\bar{p}_1 - \bar{p}_2| < \delta_\varepsilon, \quad |\bar{q}_1 - \bar{q}_2| < \delta_\varepsilon.$$

In view of the uniform continuity of the functions $u(x, y)$, $p(x, y)$ and $q(x, y)$ on R, there is another number $\delta_\varepsilon^* > 0$ (which for convenience will be chosen such

that $\varepsilon > \delta_\varepsilon > \delta_\varepsilon^* > 0$) such that

$$|u(\xi_1,\eta_1) - u(\xi_2,\eta_2)| < \delta_\varepsilon < \varepsilon,$$

$$|p(\xi_1,\eta_1) - p(\xi_2,\eta_2)| < \delta_\varepsilon < \varepsilon,$$

$$|q(\xi_1,\eta_1) - q(\xi_2,\eta_2)| < \delta_\varepsilon < \varepsilon,$$

whenever the points (ξ_1,η_1), (ξ_2,η_2) of R satisfy the inequalities

$$|\xi_1 - \xi_2| < \delta_\varepsilon^*, \quad |\eta_1 - \eta_2| < \delta_\varepsilon^*.$$

Further, there is a positive integer N_ε such that

$$|u(\xi,\eta) - u_{m_r n_r}(\xi,\eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$

$$|p(\xi,\eta) - p_{m_r n_r}(\xi,\eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$

$$|q(\xi,\eta) - q_{m_r n_r}(\xi,\eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$

and (*cf.* Section 3 for the definitions of α_m and β_n) also

$$\alpha_{m_r} < \delta_\varepsilon^* < \delta_\varepsilon, \quad \beta_{n_r} < \delta_\varepsilon^* < \delta_\varepsilon,$$

whenever

$$m_r > N_\varepsilon, \quad n_r > N_\varepsilon,$$

and (ξ,η) is any point of the rectangle R.

Let (x, y) be a point of R and m_r and n_r be positive integers such that $m_r > N_\varepsilon$ and $n_r > N_\varepsilon$. These positive integers m_r and n_r and the numbers ε, δ_ε, δ_ε^* will be supposed fixed during the immediate discussion. The notation of Section 2 (for example, writing x_k instead of $x_{m_r,k}$) will be used in the next computation for simplicity in writing. There are integers k and l, with $0 \leqq k \leqq m_r - 1$ and $0 \leqq l \leqq n_r - 1$, such that $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$, *i.e.*, such that the point (x, y) being examined lies in the closed subrectangle $R_{kl}^{m_r n_r}$. Recall that $\sigma(0) = \tau(0)$ and consider the difference

$$u_{m_r n_r}(x,y) - \left[\sigma(x) + \tau(y) - \sigma(0) + \int_{x_0}^{x}\int_{y_0}^{y} f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta\right],$$

which may be written

$$\begin{aligned} u_{m_r n_r}(x,y) - \Bigg[\sigma(x) + \tau(y) - \sigma(0) &+ \\ &+ \sum_{i=1}^{k}\sum_{j=1}^{l}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta + \\ &+ \sum_{j=1}^{l}\int_{x_k}^{x}\int_{y_{j-1}}^{y_j} f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta + \\ &+ \sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_{y_l}^{y} f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta + \\ &+ \int_{x_k}^{x}\int_{y_l}^{y} f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\,d\xi\,d\eta\Bigg]. \end{aligned}$$

Recalling the definition of $u_{m_r u_r}(x, y)$ from Section 2, and the fact that

$$f_{i-1,j-1} = f(x_{i-1}, y_{j-1}, u_{m_r n_r}(x_{i-1}, y_{j-1}), p_{m_r n_r}(x_{i-1} y_{j-1}), q_{m_r n_r}(x_{i-1}, y_{j-1}),$$

from Sections 2 and 3, one has, for example, that

$$\begin{aligned} &|f_{i-1,j-1} - f(\xi,\eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta))| \\ &\quad \leqq |f(x_{i-1}, y_{j-1}, u_{m_r n_r}(x_{i-1}, y_{j-1}), p_{m_r n_r}(x_{i-1}, y_{j-1}), q_{m_r n_r}(x_{i-1}, y_{j-1})) - \\ &\quad\quad - f(x_{i-1}, y_{j-1}, u(x_{i-1}, y_{j-1}), p(x_{i-1}, y_{j-1}), q(x_{i-1}, y_{j-1}))| + \\ &\quad\quad + |f(x_{i-1}, y_{j-1}, u(x_{i-1}, y_{j-1}), p(x_{i-1}, y_{j-1}), q(x_{i-1}, y_{j-1})) - \\ &\quad\quad - f(\xi,\eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta))| < 2\varepsilon, \end{aligned}$$

whenever $x_{i-1} \leqq \xi \leqq x_i$ and $y_{j-1} \leqq \eta \leqq y_j$. Consequently, the absolute value of the difference $u_{m_r n_r} - [\dots]$ is less than (see Section 2 for the definition of the constants C and D)

$$\begin{aligned} &|\sigma(x) - \sigma(x_k)| + |\tau(y) - \tau(y_l)| + C\cdot(x - x_k) + D\cdot(y - y_l) + \\ &\qquad + 2\varepsilon(x_k - x_0)(y_l - y_0) + 2\varepsilon(x - x_k)\Big(\sum_{j=1}^{l}(y_j - y_{j-1})\Big) + \\ &\qquad + 2\varepsilon(y - y_l)\Big(\sum_{i=1}^{k}(x_i - x_{i-1})\Big) + 2\varepsilon(x - x_k)(y - y_l) \\ &\quad \leqq 2\varepsilon(C + D + ab + \varepsilon b + \varepsilon a + \varepsilon^2) \end{aligned}$$

whenever $m_r > N_\varepsilon$ and $n_r > N_\varepsilon$, and hence

$$u(x,y) = \sigma(x) + \tau(y) - \sigma(0) + \int_{x_0}^{x}\int_{y_0}^{y} f(\xi,\eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta))\, d\xi\, d\eta$$

for any (x, y) in R. From this last equality it follows that $\partial u/\partial x$, $\partial u/\partial y$, and $\partial^2 u/\partial x\,\partial y$ $(=\partial^2 u/\partial y\,\partial x)$ exist and are continuous throughout the rectangle R. As a matter of fact

$$\frac{\partial u}{\partial x}(x,y) = \sigma'(x) + \int_{y_0}^{y} f(x,\eta, u(x,\eta), p(x,\eta), q(x,\eta))\, d\eta,$$

$$\frac{\partial u}{\partial y}(x,y) = \tau'(y) + \int_{x_0}^{x} f(\xi, y, u(\xi,y), p(\xi,y), q(\xi,y))\, d\xi,$$

while $$\frac{\partial^2 u}{\partial x\,\partial y}(x,y) = \frac{\partial^2 u}{\partial y\,\partial x}(x,y) = f(x,y,u(x,y), p(x,y), q(x,y)),$$

for any (x, y) of R.

The proofs of Theorems 2 and 3 will be complete once it is shown that $\partial u/\partial x \equiv p$ and $\partial u/\partial y \equiv q$. It suffices to consider only $\partial u/\partial x$. Let $\varepsilon > 0$ be given, and the numbers $\varepsilon > \delta_\varepsilon > \delta_\varepsilon^* > 0$ and $m_r > N_\varepsilon$, $n_r > N_\varepsilon$ be as in the argument just carried out. Let (x, y) be a point of R. There are two cases to consider: either $x_k \leqq x < x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$ for suitable integers k and l, with $0 \leqq k \leqq m_r - 2$ and $0 \leqq l \leqq n_r - 1$ or $x_{m_r-1} \leqq x \leqq x_{m_r} \equiv x_0 + a$, and $y_l \leqq y \leqq y_{l+1}$ with $0 \leqq l \leqq n_r - 1$. Consider the difference

$$p_{m_r n_r}(x,y) - \Big[\sigma'(x) + \int_{y_0}^{y} f(x,\eta, u(x,\eta), p(x,\eta), q(x,\eta))\, d\eta\Big],$$

which may be written (in either of the two cases mentioned, with $k=m_r-1$ in the second case)

$$p_{m_r n_r}(x,y) - \Big[\sigma'(x) + \sum_{i-1}^{l} \int_{y_{j-1}}^{y_j} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta + \\ + \int_{y_l}^{y} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta\Big].$$

Recalling the definition of $p_{m_r n_r}(x, y)$ from Section 5, and the fact that from Sections 2 and 3

$$f_{i-1,j-1} = f\big(x_{i-1}, y_{j-1}, u_{m_r n_r}(x_{i-1} y_{j-1}), p_{m_r n_r}(x_{i-1}, y_{j-1}), q_{m_r n_r}(x_{i-1}, y_{j-1})\big),$$

one has again, for example, that

$$\big|f_{i-1,j-1} - f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))\big| < 2\varepsilon,$$

whenever $x_{i-1} \leqq \xi \leqq x_i$ and $y_{j-1} \leqq \eta \leqq y_j$. Besides, the mean value theorem of the differential calculus and the definition of the constant C of Section 2 imply that

$$\left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \sigma'(x)\right| = |\sigma'(x^*) - \sigma'(x)| \leqq C\,|x^* - x| \leqq C\,\alpha_{n_r}.$$

Consequently, the absolute value of the difference $p_{m_r n_r}(x, y) - [\ldots]$ is less than

$$\varepsilon C + 2\varepsilon\Big(\sum_{i=1}^{k}(x_i - x_{i-1})\Big) + 2\varepsilon(x - x_k) \leqq \varepsilon(C + 2a),$$

whenever $m_r > N_\varepsilon$ and $n_r > N_\varepsilon$, and hence

$$p(x,y) = \sigma'(x) + \int_{y_0}^{y} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta$$

for any (x, y) in R. Since the right hand side of the last equation is already known to be equal to $\frac{\partial u}{\partial x}(x, y)$, it follows that $\frac{\partial u}{\partial x} = p$, as desired. By symmetry one has also that

$$q(x,y) = \tau'(y) + \int_{x_0}^{x} f\big(\xi,y,u(\xi,y),p(\xi,y),q(\xi,y)\big)\,d\xi$$

for any (x, y) in R, from which it follows that $\partial u/\partial y = q$, and the proof is complete.

Under the hypotheses of Theorem 3, the preceding argument shows that any singly infinite subsequence $\{u_{m_r n_r}(x, y)\}$, where $\lim_{r\to\infty} m_r = \infty$ and $\lim_{r\to\infty} n_r = \infty$, contains a subsequence which converges uniformly on R to a solution. On the other hand, under the hypotheses of Theorem 2 (in which case there is but one solution) the preceding argument implies that the whole double sequence $\{u_{mn}(x, y)\}$ converges to the solution, *i.e.* that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} u_{mn}(x, y)$$

is the solution, the convergence being uniform on R.

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